TOKYO J. MATH.
VOL. 17, NO. 2, 1994

Elementary Functions Based on Elliptic Curves

Keiji NISHIOKA

Keio University

1. w-elementary functions.

Let K be an ordinary differential field of characteristic 0 with a single differentiation D. We assume K has the field of constants C which is algebraically closed. We fix a universal extension U of K.

Let R be a differential field extension of K with finite transcendence degree over K. By $\Omega_{R/K}$ denote the R-module of all differentials of R over K and by $d_{R/K}$ the canonical derivation of R to $\Omega_{R/K}$. With D there associates an additive endomorphism D^1 of $\Omega_{R/K}$, satisfying

$$D^1(ad_{R/K}b)=D(a)d_{R/K}b+ad_{R/K}(Db), \qquad a, b\in R\,.$$

If a and b are algebraically dependent over C then $D^1(ad_{R/K}b)=d_{R/K}(aDb)$ holds (cf. Rosenlicht [3]).

Consider an elliptic curve E defined over C which is given in the Weierstrass form

$$y^2=4x^3-g_2x-g_3\,, \qquad g_2, g_3\in C\,, \quad g_2^3-27g_3^2\neq 0\,.$$

By $E(R)$ we denote the set of all R-rational points on E and by $E(R)^*$ the set $E(R)\setminus\{O_E\}$, where O_E denotes the zero element of E. Let $P=(1, x, y)$ be an R-rational point on E. We consider the following three types of peculiar differentials in $\Omega_{R/C}$

$$\omega_I(P)=\frac{d_{R/C}x}{y} \quad\text{and}\quad \omega_{II}(P)=x\frac{d_{R/C}x}{y}\,.$$

For a point $Q=(1, x(Q), y(Q))\in E(C)^*$ by $\omega_{III}(P\,;Q)$ we denote the differential in $\Omega_{R/C}$

$$\frac{1}{2}\,\frac{y+y(Q)}{x-x(Q)}\,\frac{d_{R/C}x}{y}\,.$$

In case $P\in E(C)$ we think that these differentials take the zero differential. In the case where $R=C(x, y)$ with $x\notin C$ any differential ω in $\Omega_{R/C}$ has the unique expression

Received April 20, 1993