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The Hasse Norm Principle for the Maximal Real Subfields of Cyclotomic Fields

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§1. Introduction.

Let K/k be a finite extension of number fields. Let J_K be the idele group of K and $N_{K/k}$ the norm map from K to k. The group $N_{K/k}K^\times$ of global norms is a subgroup of finite index in $k^\times \cap N_{K/k}J_K$. We say that the Hasse norm principle (abbreviated to HNP) holds for K/k if $k^\times \cap N_{K/k}J_K = N_{K/k}K^\times$. We simply say that HNP holds for K if HNP holds for $K/\boldsymbol{Q}$. The classical Hasse norm theorem asserts that if K/k is a cyclic extension, then HNP holds for K/k.

Several authors have studied the validity of HNP for abelian extensions. In [3] and [4], Gerth and Gurak independently gave necessary and sufficient conditions for HNP to hold for $\boldsymbol{Q}(\zeta_m)$, where $m \not\equiv 2 \pmod 4$ is a positive integer and ζ_m is a primitive m-th root of unity. If HNP holds for $\boldsymbol{Q}(\zeta_m)$, then it holds also for its maximal real subfield $\boldsymbol{Q}(\zeta_m)^+$ (Proposition 1 below). However, the converse is not always true. In this paper, we will give a necessary and sufficient condition for HNP to hold for $\boldsymbol{Q}(\zeta_m)^+$.

§2. Theorems.

Let $m \not\equiv 2 \pmod 4$ be a positive integer, and let p_1, p_2, p_3 and p_4 be distinct odd primes, and e, a_1, a_2, a_3, a_4 non-negative integers. We denote by $\left(\frac{*}{*}\right)$ the Legendre symbol and define ε_i and $\varepsilon_{i,j}$ ($\in \{0, 1\}$) by $(-1)^{\varepsilon_i} = \left(\frac{2}{p_i}\right)$ and $(-1)^{\varepsilon_{i,j}} = \left(\frac{p_j}{p_i}\right)$, respectively.

(A) Suppose that m has at most three distinct prime divisors and that $m \neq 2^e p_1^{a_1} p_2^{a_2}$, $e \geq 3$. In this case, we know necessary and sufficient conditions for HNP to hold for $\boldsymbol{Q}(\zeta_m)$ (cf. [3, 4]).

THEOREM 1. *HNP does not hold for $\boldsymbol{Q}(\zeta_m)$ but does hold for $\boldsymbol{Q}(\zeta_m)^+$ if and only if*

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