TOKYO J. MATH.
VOL. 18, NO. 1, 1995

Large Time Behavior of Solution for Hartree Equation with Long Range Interaction

Hitoshi HIRATA

University of Tokyo

(Communicated by S.T. Kuroda)

§1. Introduction and theorem.

In this paper, we study the asymptotic behavior as $t\to\infty$ of the solutions of time dependent Hartree equations

$$i\partial_t u=-\frac{1}{2}\Delta_x u+(|x|^{-\gamma}*|u|^2)u \qquad (H_\gamma)$$

for $\gamma\leq 1$, where $u=u(t,x)$, $(t,x)\in\boldsymbol{R}\times\boldsymbol{R}^n$. We write $\|\cdot\|_p$ for L^p-norm, $(\cdot,\cdot)$ for L^2-coupling, $H^{l,k}=\{u\in L^2:\sum_{|\alpha|\leq l}\|\partial_x^\alpha u\|_2+\sum_{|\beta|\leq k}\|x^\beta u\|_2<\infty\}$ for $l,k=0,1,2,\cdots$ and $U(t)=\exp[(i/2)t\Delta_x]$. There is a large body of literature on the equation (H_γ). It is well-known that a unique global solution exists for any $u_0\in H^{1,0}$ if $0\leq\gamma<\min\{4,n\}$. (cf. [GV], [DF] etc.) If we assume $\gamma>1$, any above solution u behaves like free solution as t goes to infinity: that is, there exists an asymptotic state u_+ such that $\|u(t)-U(t)u_+\|_X\to 0$ as $t\to\infty$ in a suitable space X. On the other hand, if $\gamma\leq 1$, no non-trivial solution becomes asymptotically free. (See e.g. [G], [HT], [NO] etc.) But inferring on the analogy of linear long range scattering theory, the solution of this case is expected to behave *almost free*. That is, if we slightly modify the solution u by a certain phase S, then this modified solution is expected to become asymptotically free. Following result for the case $n\geq 3$ suggests above expectation.

THEOREM 1. *Let $u_0\in H^{1,1}$, $1\geq\gamma>2/3$ if $n\geq 4$, $1\geq\gamma>(\sqrt{17}-1)/4$ if $n=3$, and $u(t,x)$ be a solution of (H_γ) such that $u(0,x)=u_0(x)$. If we put*

$$S(\tau,\xi):=\int_1^\tau(|x|^{-\gamma}*|u|^2)(s,s\xi)ds=\int_1^\tau\int_{\boldsymbol{R}^n}\frac{|u(s,y)|^2}{|s\xi-y|^\gamma}dyds\,, \qquad (1)$$

then $u_+:=\text{w-lim}_{t\to\infty}M(t)U(-t)\exp[iS(t,t^{-1}\cdot)]u(t,\cdot)$ *exists in* $H^{1,0}$. *Here* $M(t)=\exp[(i/(2t))|x|^2]$.

Received November 27, 1993