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# Direct Sum Decomposition of the Integers

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## §1. Introduction.

In this paper we shall be concerned with the direct sum decomposition $\boldsymbol{Z}=A\oplus B$ of the set $\boldsymbol{Z}$ of all integers, where both subsets $A$ and $B$ are infinite subsets. Recently, a very interesting connection between such a decomposition of $\boldsymbol{Z}$ (and of $\boldsymbol{N}$) and properties of infinite measure preserving ergodic transformations was discovered, and exploiting this connection, a number of significant results have been obtained characterizing the nature of the summands that appear in such a decomposition, see [2], [3], [4] and [5].

While it is well-known and is not difficult to characterize the infinite subsets that appear as direct summands of the decomposition $\boldsymbol{N}=A\oplus B$, see, for example [1], [6], the situation is very different for the case of the direct sum decomposition of $\boldsymbol{Z}$, where it seems to be very difficult to give a reasonable characterization of summands in general, see Proposition 2.2 below. On the other hand, if one fixes one of the summands of such a decomposition to be a "reasonable set" in some sense, then one can give some interesting characterizations for infinite subsets of $\boldsymbol{Z}$ that can be a complement of this set in the decomposition of $\boldsymbol{Z}$. If we let the set $A$ to be one of the sets that appear as a direct summand of the decomposition of $\boldsymbol{N}$, for example, it follows from the known result mentioned above that there exists a unique subset $B$ such that $\boldsymbol{N}=A\oplus B$ and it is not difficult to show that for this $B$, $A\oplus(-B)=\boldsymbol{Z}$ holds, and furthermore, one can construct by starting with this $B$ many other complements of $A$ in $\boldsymbol{Z}$, see Proposition 2.3 below. In fact, in [3] and [4], it was shown by using ergodic theory that such a set $A$ always has uncountably many distinct complements in $\boldsymbol{Z}$, some of which can be of very different nature from those described in Proposition 2.3.

So, in this paper, we shall take $A$ to be one of the sets that can appear as a direct summand of the decomposition of $\boldsymbol{N}$; in fact, for the sake of simplicity, we take $A$ to be the simplest of such sets, namely, let $A$ consist of 0 and all finite sums of distinct odd powers of 2, and give a characterization of sets $C$ that can appear as a complement of this $A$ in the direct sum decomposition of $\boldsymbol{Z}$. For this set $A$, S. Eigen, A. Hajian and

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