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On the Variance of the Feasible Weighted Least Squares Estimator

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1. Introduction.

Let X_{ij} $(j=1, 2, \cdots, n_i)$ be mutually independent random variables distributed according to $N(\theta, \sigma_i^2)$ $(i=1, \cdots, k)$, where θ and $\{\sigma_i^2\}_{i=1}^k$ are unknown parameters. We consider an unbiased estimator of θ defined by

$$(1.1) \qquad \hat{\theta}_k=\left\{\sum_{i=1}^{k} c_i n_i \hat{\sigma}_i^{-2} \bar{X}_i\right\}\left\{\sum_{i=1}^{k} c_i n_i \hat{\sigma}_i^{-2}\right\}^{-1},$$

where $\bar{X}_i=\sum_{j=1}^{n_i} X_{ij}/n_i$, $\hat{\sigma}_i^2=\sum_{j=1}^{n_i}(X_{ij}-\bar{X}_i)^2/(n_i-1)$ $(i=1, \cdots, k)$ and $\{c_i\}_{i=1}^k$ is a sequence of positive constants.

For $k\geq 3$ and $n_i\geq 6$ $(i=1, \cdots, k)$, Shinozaki (1978) proved that $V[\hat{\theta}_k]\leq \min_{1\leq i\leq k} V[\bar{X}_i]$ holds if and only if $c_q c_p^{-1}\leq 2(n_p-1)(n_q-5)(n_p+1)^{-1}(n_q-1)^{-1}$ for any $p\neq q$. For example, if we choose $c_i=(n_i-5)(n_i-1)^{-1}$ $(i=1, \cdots, k)$, this condition is equivalent to $n_i\geq 11$ $(i=1, \cdots, k)$. It follows from this that the combined estimator $\hat{\theta}_k$ is preferable to each $\bar{X}_i$ when $c_i=(n_i-5)(n_i-1)^{-1}$ and $n_i\geq 11$ $(i=1, \cdots, k)$.

Now we consider the accuracy of $\hat{\theta}_k$ when $n_i\geq 6$ $(i=1, \cdots, k)$. Though we would like to evaluate the variance of $\hat{\theta}_k$, it seems to be difficult to obtain its exact expression. When $k\to +\infty$, however, Takeuchi (1994) indicated that the ratio of $V[\hat{\theta}_k]$ to the Cramér-Rao lower bound $m^{-1}\{\sum_{i=1}^k \sigma_i^{-2}\}^{-1}$ is greater than or equal to $(m-3)(m-5)^{-1}$ when $n_i=m\geq 6$ $(i=1, \cdots, k)$.

In this paper, the limiting variance of $\hat{\theta}_k$-type estimator is obtained. The lists of notations and conditions are given in Section 2.1. In Section 2.2, the asymptotic properties of $V[\hat{\theta}_k]$ are described. Theorem 2.1 asserts that the ratio of $V[\hat{\theta}_k]$ to the Cramér-Rao lower bound satisfies some limit relation. Theorem 2.2 ensures the existence of the limiting variance of $\hat{\theta}_k$ and Theorem 2.3 gives the optimal estimator in the sense that it attains the minimum of the limiting variance in some class of estimators. In Section 2.3, three auxiliary lemmas are proved. The proofs of Theorem 2.1–Theorem

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