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Dipolarizations in Compact Lie Algebras and Homogeneous Parakähler Manifolds

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Introduction.

Let $\mathfrak{g}$ be a real Lie algebra and $\mathfrak{g}^{\pm}$ be two subalgebras of $\mathfrak{g}$ and ρ be an alternating 2-form on $\mathfrak{g}$. Then the triple $\{\mathfrak{g}^+, \mathfrak{g}^-, \rho\}$ is called a *weak dipolarization* in $\mathfrak{g}$ if the following conditions are satisfied:

(WD1) $\mathfrak{g}=\mathfrak{g}^+ + \mathfrak{g}^-$,

(WD2) $\rho(\mathfrak{g}^+, \mathfrak{g}^+)=\rho(\mathfrak{g}^-, \mathfrak{g}^-)=0$,

(WD3) $\rho(X, \mathfrak{g})=0$ if and only if $X\in\mathfrak{g}^+ \cap \mathfrak{g}^-$,

(WD4) $\rho([X, Y], Z)+\rho([Y, Z], X)+\rho([Z, X], Y)=0$, $\forall X, Y, Z\in\mathfrak{g}$.

A dipolarization in $\mathfrak{g}$ is a triple $\{\mathfrak{g}^+, \mathfrak{g}^-, f\}$, formed by two subalgebras $\mathfrak{g}^{\pm}$ and a linear form f, which satisfies the following conditions:

(D1) $\mathfrak{g}=\mathfrak{g}^+ + \mathfrak{g}^-$,

(D2) $f([\mathfrak{g}^+, \mathfrak{g}^+])=f([\mathfrak{g}^-, \mathfrak{g}^-])=0$,

(D3) $f([X, \mathfrak{g}])=0$ if and only if $X\in\mathfrak{g}^+ \cap \mathfrak{g}^-$.

A dipolarization $\{\mathfrak{g}^+, \mathfrak{g}^-, f\}$ is itself a weak dipolarization, since df satisfies (WD2)–(WD4). A weak dipolarization is called *symmetric* if $\mathfrak{g}^+$ is Lie-isomorphic to $\mathfrak{g}^-$. Otherwise it is called *nonsymmetric*. A dipolarization (resp. weak dipolarization) is called *trivial*, if $\mathfrak{g}^+=\mathfrak{g}^-=\mathfrak{g}$, and if $f=0$ (resp. $\rho=0$).

The notions of dipolarizations and weak dipolarizations in a Lie algebra were first introduced by Kaneyuki ([6]) to describe a class of homogeneous symplectic manifolds, called *homogeneous parakähler manifolds*. Let us recall the definition of homogeneous parakähler manifolds ([6]). A parakähler manifold M is, by definition, a symplectic manifold which admits a pair of transversal Lagrangian foliations. If a Lie group G acts on M as symplectomorphisms which preserves each of the two foliations, then we say that the parakähler structure is G-invariant. Furthermore, if G acts transitively on M, then M is said to be a *homogeneous parakähler manifold*. It was proved in [6] that a necessary and sufficient condition for the existence of an invariant parakähler structure on $M=G/H$ (H is an isotropy subgroup) is that there exists a weak dipolarization in

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