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On Direct Sum Decomposition of Integers and Y. Ito's Conjecture

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Let $\mathbf{N}=\{0, 1, 2, \cdots\}$. Every element $a\in\mathbf{N}$ can be expressed as

$$a=\sum_{i=0}^{n}\alpha_i 2^i \qquad \text{for some } n$$

where $\alpha_i\in\{0, 1\}$ for all i's.

We can identify a with $(\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_n, 0, 0, \cdots)\in\{0, 1\}^{\mathbf{N}}$. We also identify $\{0, 1\}^{\mathbf{N}}$ in the usual way with $\mathbf{Z}_2$, the completion of $\mathbf{Z}$ in the 2-adic valuation norm.

Thus $\mathbf{N}$ is imbedded in $\mathbf{Z}_2$ as the 0-1-sequences with only finitely many 1's, while the negative integers are imbedded as those with finitely many 0's. For example, if a is a positive integer corresponding to the 0-1-sequence as above with $\alpha_i=0$ for any $i>n$, then $-a$ is identified with $(\underbrace{0, \cdots, 0, 1}_{m+1}, \overline{\alpha_{m+1}}, \overline{\alpha_{m+2}}, \cdots)$, where m is the smallest i with $a_i=1$ and we denote $\overline{0}=1, \overline{1}=0$.

We denote by $\bar{E}$ the closure of a subset E of $\mathbf{Z}_2$.

Let us denote

$$A=\left\{\sum_i \varepsilon_i 2^{2i+1}\ ;\ \varepsilon_i\in\{0, 1\} \text{ and } \varepsilon_i=1 \text{ for finitely many } i\text{'s}\right\}.$$

For $\omega=(\omega_0, \omega_1, \cdots)\in\{-1, 1\}^{\mathbf{N}}$ with $\omega_i=-1$ for infinitely many times, denote

$$B_\omega=\left\{\sum_i \varepsilon_i\omega_i 2^{2i}\ ;\ \varepsilon_i\in\{0, 1\} \text{ and } \varepsilon_i=1 \text{ for finitely many } i\text{'s}\right\}.$$

Let us denote

$$\mathscr{C}(A)=\{C\subset\mathbf{Z};\ 0\in C \text{ and } A\oplus C=\mathbf{Z}\}\,,$$

where $A\oplus C=\mathbf{Z}$ implies that any element in $\mathbf{Z}$ can be written uniquely as a sum of elements in A and C.

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