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Critical Metrics of the Scalar Curvature Functional

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1. Introduction.

Let M be a compact connected n-manifold and $\mathcal{M}(M)$ the space of Riemannian metrics on M. We study the critical metrics of the following functional;

$$\mathcal{S}^p : \mathcal{M}(M) \to \mathbf{R} ; \qquad g \mapsto \frac{\int_M R_g^p dv_g}{(\int_M dv_g)^{(n-2p)/n}} ,$$

where R_g is the scalar curvature of g and $p \in \mathbf{N}$.

The first variation formula for $\mathcal{S}^p$ is

$$\nabla^2 R_g^{p-1} = \frac{1}{n} \Delta R_g^{p-1} g + R_g^{p-1}\left(\mathrm{Ric}_g - \frac{R_g}{n} g\right), \tag{i}$$

where Ric_g is the Ricci tensor of g. Taking the divergence of (i) with respect to g, we have

$$\Delta R_g^{p-1} = \frac{n-2p}{2p(n-1)}(R_g^p - \overline{R_g^p}) , \tag{ii}$$

where $\overline{R_g^p} = \int_M R_g^p dv_g / \int_M dv_g$. The equation (ii) is also the first variation formula for $\mathcal{S}^p|_C$, where C is a conformal class of $\mathcal{M}(M)$.

Obviously, if $R_g \equiv 0$ or g is an Einstein metric, the metric g satisfies the equation (i). A metric of constant scalar curvature satisfies the equation (ii). The question is whether the converses are true or not.

The case $p=1$ is well-known (e.g. [5]). When $n=2$, $\mathcal{S}^2|_C$ was studied by Calabi ([4], see also Section 3). If $n \geq 4$ and $p=n/2$, the answer is positive (e.g. [2]). If $n=3$, $p=2$ and R_g does not change the sign, then the metric which satisfies (i) is of constant scalar curvature ([1]). According to Anderson ([1]), the general case is an open question.

In this paper, we show the following results which are extentions of Anderson's

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