TOKYO J. MATH.
VOL. 22, NO. 1, 1999

Substitution Invariant Inhomogeneous Beatty Sequences

Takao KOMATSU

Nagaoka National College of Technology

(Communicated by K. Shinoda)

1. Introduction.

Given a real irrational θ and an arbitrary real ϕ we get

$$f_n = f(n; \theta, \phi) = [(n+1)\theta + \phi] - [n\theta + \phi] - [\theta] .$$

For brevity we write the infinite sequence (f_n) as $f_{\theta,\phi} = f_1 f_2 f_3 \cdots$. Here our purpose will be to find those substitutions W leaving $f_{\theta,\phi}$ invariant, that is, so that $W(f_{\theta,\phi}) = f_{\theta,\phi}$; and the θ and ϕ admitting such a substitution. We recall that a substitution is a pair of maps

$$W: 0 \longrightarrow W_0 , \qquad 1 \longrightarrow W_1 ,$$

where W_0 and W_1 are finite strings of 0's and 1's. We considered the homogeneous case $\phi = 0$ in [5]. In this paper we shall give some solutions appropriate to the inhomogeneous case $\phi \neq 0$. The similar problem has been discussed by Ito and Yasutomi [3] and by Crisp [1], but we provide rather different argument.

Let $\theta = [a_0, a_1, a_2, \cdots]$ denote the continued fraction expansion of θ, where

$$\theta = a_0 + \theta_0 , \qquad a_0 = [\theta] ,$$
$$1/\theta_{n-1} = a_n + \theta_n , \qquad a_n = [1/\theta_{n-1}] \quad (n = 1, 2, \cdots) .$$

The n-th convergent $p_n/q_n = [a_0, a_1, \cdots, a_n]$ of θ is given by the recurrence relations

$$p_n = a_n p_{n-1} + p_{n-2} \quad (n = 0, 1, \cdots) , \quad p_{-2} = 0 , \quad p_{-1} = 1 ,$$
$$q_n = a_n q_{n-1} + q_{n-2} \quad (n = 0, 1, \cdots) , \quad q_{-2} = 1 , \quad q_{-1} = 0 .$$

Further, let $\phi = {}_\theta[b_0, b_1, b_2, \cdots,]$ be the expansion of ϕ in terms of the sequence $\{\theta_0, \theta_1, \cdots\}$, where

$$\phi = b_0 - \phi_0 , \qquad b_0 = \lceil \phi \rceil ,$$
$$\phi_{n-1}/\theta_{n-1} = b_n - \phi_n , \qquad b_n = \lceil \phi_{n-1}/\theta_{n-1} \rceil \quad (n = 1, 2, \cdots) .$$

Received August 27, 1997
Revised April 27, 1998