TOKYO J. MATH.
VOL. 22, NO. 1, 1999

The Involutions of Compact Symmetric Spaces, IV

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Introduction.

In this part, we will discuss and determine the signature $\tau(M)$ and the g-signature g-$\tau(M)$ (See [AS]) of every compact oriented symmetric space M, (making the description of $\tau(M)$ in [N88] intelligible, in particular) as well as self-intersections $SI(N;M)$ of subspaces N in M which are beautifully related to g-$\tau(M)$ by the g-signature theorem or the generalized Lefschetz fixed point theorem of Atiyah-Singer [AS] in case g is an orientation-preserving involution of M. An example of geometric applications will be added.

Our method is basically to apply the Atiyah-Singer theory [AS], especially the g-signature theorem (1.3) below to our geometric results ([N88] and others). We have determined those invariants for each space in a few ways. The value of $\tau(M)$ was stated in [N88] with a very brief explanation (involving a careless mistake), but we will give a more detailed proof. Informations on g-$\tau(M)$ and on $SI(N;M)$ are reciprocal to some extent. The self-intersection $SI(N;M)=[B]$ is realized by a (symmetric) subspace B of M (in the cases discussed in this paper).

0.1 THEOREM ([N88], 10.1). *If the signature $\tau(M)$ $(\geqq 0)$ is positive for a simple 1-connected M, then $\tau(M)$ equals the indicated value below*: $\frac{1}{2}\tau(G_{2p}^{o}(\mathbf{R}^{2n}))=\tau(G_p(\mathbf{C}^n))=\tau(G_p(\mathbf{H}^n))=\chi(G_p(\mathbf{R}^n))$, *the Euler number of* $G_p(\mathbf{R}^n)$, $\tau(\mathrm{EII})=4$, $\tau(\mathrm{EIII})=3$, $\tau(\mathrm{EVI})=7=\tau(\mathrm{EVIII})$, $\tau(\mathrm{EIX})=8$, *and* $\tau(\mathrm{FII})=1=\tau$ (GI). Here the symbols for the symmetric spaces are Cartan's [H] with a few exceptions such as $G_p(V)$ meaning the Grassmann manifold of the p dimensional subspaces of a vector space V and $G_p^o(\mathbf{R}^n)$ which means that of the oriented p-subspaces of $\mathbf{R}^n$; $G_p^o(\mathbf{R}^n)$ is 1-connected except that it consists of two points for $p=0$ or n and $G_1^o(\mathbf{R}^2)$ is a circle. The known Euler number $\chi(G_p(\mathbf{R}^n))$ equals the binomial coefficient ${}_{[n/2]}C_{[p/2]}$ *if* $p(n-p)=\dim G_p(\mathbf{R}^n)$ is even and 0 otherwise.

0.2 COROLLARY. *One has*

$$(0\leqq)3\tau(M)\leqq\chi(M) \qquad \textit{if } \dim M>0\,,$$

and this is sharp. In particular, the equality

Received October 7, 1997