TOKYO J. MATH.
VOL. 22, NO. 1, 1999

Substitution in Two Symbols and Transcendence

Kumiko NISHIOKA, Taka-aki TANAKA and Zhi-Ying WEN

Keio University and Wuhan University

1. Introduction.

Let $A=\{a_1, \cdots, a_n\}$ be a finite nonempty set of symbols and let A^* and A^ω denote the sets of all finite words over A and all sequences $x_0x_1\cdots x_k\cdots$ $(x_k\in A)$, respectively. Let λ be the empty word. A *substitution* (over A) is a map $\sigma: A\to A^*\setminus\{\lambda\}$, which has a natural extension to $\Omega=A^*\cup A^\omega$ by concatenation: $\sigma(x_0x_1\cdots)=\sigma(x_0)\sigma(x_1)\cdots$. If a_i is a prefix of $\sigma(a_i)$ and the length of $\sigma(a_i)$ is greater than 1, then there is a unique $w\in\Omega$ having a prefix a_i and being a fixed point of σ, which means that $\sigma(w)=w$. Any real algebraic irrational θ can be uniquely expressed as

$$\theta=\sum_{k=-m}^{\infty}\varepsilon_k 2^{-k}, \tag{1}$$

where m is a nonnegative integer depending on θ and $\varepsilon_k=0$ or 1. The problem we are interested in is whether the sequence $\varepsilon_0\varepsilon_1\cdots\in\{0, 1\}^\omega$ is a fixed point of any substitution over $\{0, 1\}$ or not.

Generally, for a fixed point $w=x_0x_1\cdots$ of the given substitution σ, we define the generating function of w for a_i by

$$f_i(z)=\sum_{k=0}^{\infty}\chi_k(w; a_i)z^k, \tag{2}$$

where $\chi_k(w; a_i)=1$ if $x_k=a_i$, and otherwise $\chi_k(w; a_i)=0$, so that

$$\sum_{i=1}^{n} f_i(z)=\sum_{k=0}^{\infty} z^k=\frac{1}{1-z}.$$

It is known that $f_i(z)$ $(1\le i\le n)$ satisfy a Mahler type functional equation if σ is of constant length, which means that each $\sigma(a_i)$ $(1\le i\le n)$ has the same length ≥ 2, and it is also known that if σ is of nonconstant length, i.e., the lengths of $\sigma(a_i)$ $(1\le i\le n)$ are not equal, then we can construct $g_1(\mathbf{z}), \cdots, g_n(\mathbf{z})\in\mathbf{Q}[[z_1, \cdots, z_n]]$ satisfying a Mahler

Received July 4, 1997
Revised December 12, 1997