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Extremal Elliptic Fibrations and Singular $K3$ Surfaces

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In memory of Professor N. Sasakura

1. Introduction.

Let $f: X \to \mathbf{P}^1$ be a relatively minimal elliptic fibration having a section (so-called Jacobian fibration) over an algebraically closed field k of characteristic $p \geq 0$. It is well known that for such a non-trivial fibration in characteristic $\neq 2, 3$ the number s of singular fibres is at least 2, and if f is non-isotrivial, then s is at least 3. In characteristic zero a complete list of non-trivial fibrations over $\mathbf{P}^1$ with three or fewer singular fibres together with Kodaira fibre types was given by U. Schmickler-Hirzebruch ([Sc-H]). Her method used the monodromy actions around critical points *à la Kodaira*. It turned out that such a surface is either a rational, or a $K3$ surface. The case of $K3$ surfaces is of interest because of several reasons. In [N1] we discussed a different approach which, as the reader can see easily, is applicable for a similar problem of classifying such fibrations in positive characteristics. *A priori* up to the action of the absolute Frobenius the classification in characteristic $p \neq 2, 3$ should be the same as in characteristic zero. In fact an essentially new idea is to involve the Kodaira-Spencer class, especially the so-called characteristic p function field analogue of Szpiro's conjecture, and the well-known theory of Ogg-Shafarevich (*cf.* [N2]). In this note we first recover the list of [Sc-H] by means of the approach mentioned above (Theorem 2.10 and the first part of theorem 3.4). Next a question arising here is to determine fo which p the Weierstrass equation of a $K3$ surface in the given classification defines a supersingular, and hence (*a priori* modulo Artin's conjecture) unirational, $K3$ surface. To this end it is natural to use the works [P-S] and [In-S] since in characteristic zero $K3$ surfaces with three singular fibres are singular in the sense of [P-S] and [In-S] (the second part of Theorem 3.4). In a sense the note may be thought as a prelude to a complete classification of elliptic pencils with three or fewer singular fibres in positive characteristics (at least,

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