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Birational Geometry of Plane Curves

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1. Introduction.

The purpose of this paper is to study curves on rational surfaces from the viewpoint of birational geometry. We begin by recalling basic notions and elementary results of birational geometry of plane curves (see [4], [5] and [8]).

Let C be a curve on a surface S. Here by curves and surfaces we mean projective irreducible varieties of dimension 1 and 2, respectively, which are defined over an algebraically closed field of characteristic zero. We shall study such pairs (S, C). Two pairs (S, C) and (S_1, C_1) are said to be *birationally equivalent* if there exists a birational map $f: S \to S_1$ such that the proper image $f[C]$ of C by f coincides with C_1. Here the proper image $f[C]$ is by definition the closure of the image $f(x)$ of the generic point x of C. When there is no danger of confusion, we say that C is birationally equivalent to C_1 as imbedded curves if two pairs (S, C) and (S_1, C_1) are birationally equivalent. A pair (W, D) is said to be *non-singular*, if both W and D are non-singular. In this case, we have complete linear systems $|m(D+K_W)|$ for any $m>0$, where K_W indicates a canonical divisor of W. The dimension $\dim|m(D+K_W)|+1$ depends on both D and W. But to simplify the notation, we use the symbol $P_m[D]$ to denote $\dim|m(D+K_W)|+1$. From this we define the *Kodaira dimension* $\kappa[D]$ of (W, D) to be the degree of $P_m[D]$ as a function in m. It is easy to see that $P_m[D]$ and $\kappa[D]$ are birational invariants of (W, D) in the above sense. In general, for $n \geqq m$, the dimensions $\dim|mD+nK_W|$ are also birational invariants. To verify this, let $h: V \to W$ be a birational morphism where both V and W are non-singular. We assume that D is non-singular and the proper inverse image of D by h, denoted by D_1, is also non-singular. Then we have

$$mD_1+nK_V \sim h^*(mD+nK_W)+mR'_h+(n-m)R_h\,,$$

where R'_h is the logarithmic ramification divisor and R_h is the ramification divisor (see [4]). Here the symbol $\sim$ denotes the linear equivalence between divisors. These divisors are effective and the images of these by h are finite sets of points. Hence,

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