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The Iwasawa λ-invariants of Real Abelian Fields and the Cyclotomic Elements

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0. Introduction.

Our aim in this paper is to introduce an element

$$c_{l,r} \in \mathbf{F}_l \quad (l: \text{a prime number},\ r: \text{an integer} > 1)$$

in a finite field $\mathbf{F}_l$, and also an element $c^{\chi}_{l,r,p^n} \in \mathbf{F}_l$ (p: some prime number, n: a positive integer) for a Dirichlet character χ. We also introduce a function $r \mapsto n(r)$ ($r > 1$, $n(r) \in \mathbf{Z}_{>0}$) where $n(r)$ is the number defined as an "index" of $c_{l,r}$'s for all l, and also introduce a function $r \mapsto n^{\chi}(r)_p$ where $n^{\chi}(r)_p$ is an "index" of c^{χ}_{l,r,p^n}. These functions $n(r)$, $n^{\chi}(r)_p$, and elements $c_{l,r}$, c^{χ}_{l,r,p^n} are defined only in terms of elementary number theory (§1, §2).

Theoretically $n(r)$ is the order of the Tate Shafarevich group of the motive $\mathbf{Z}(r)$ in the sense of [1] (and $n^{\chi}(r)$ corresponds to the χ-part of the Tate Shafarevich group of $\mathbf{Z}(r)$). In §3 we describe the relation between the elements in §1, §2 and the cyclotomic elements of Deligne and Soulé [2] [15] [16]. These cyclotomic elements (and their indexes) are useful for the arithmetic of cyclotomic fields. In this paper, we show that they can be used for a numerical verification of Greenberg's conjecture. For a totally real number field K and a prime number p, Greenberg's conjecture asserts that the Galois group of the maximal unramified abelian pro-p extension of the cyclotomic $\mathbf{Z}_p$-extension K^{cycl}_{∞} is finite [6]. When we check this conjecture numerically, one of the problems lay in studying the group of units. For example, it is difficult to find fundamental units in general. Kraft and Schoof [11] and Ichimura and Sumida [7] [8] found good criterions to verify this conjecture for real abelian number fields, in which they do not use fundamental units, but use only cyclotomic units. This paper is in the same stream and we do not use fundamental units either.

In §5, we give some simple criterions (Theorems 5.4, 5.8, cf. also Theorems 1.6, 2.5) on Greenberg's conjecture in some simple cases, and give some examples (Example 5.5 treats some quadratic fields with $p=3$ and Example 5.10 treats $K=\mathbf{Q}(\sqrt{m}, \cos(2\pi/7))$

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