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On Additive Volume Invariants of Riemannian Manifolds

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Introduction.

Let (M, g) be an n-dimensional Riemannian C^ω manifold and $p \in M$ a point. For small $r > 0$, we denote by $V_p(r)$ the volume of the geodesic ball of radius r with the center p. It is known that $V_p(r)$ is given by a power series expansion

$$V_p(r) = V_0(r)(1 + B_2(p)r^2 + B_4(p)r^4 + \cdots + B_{2k}(p)r^{2k} + \cdots)$$

where $V_0(r)$ is the volume of an n-dimensional Euclidean ball of the same radius and B_2, $B_4, \cdots, B_{2k}, \cdots$ are *the volume invariants*, which are analytic functions of p, or, more specifically, scalar curvature invariants of order $2, 4, \cdots, 2k, \cdots$ respectively (see e.g. [G1]). If (M, g) is flat, $B_{2k} \equiv 0$ for all $k \in \mathbf{N}$; we have the following conjecture:

VOLUME CONJECTURE [Gray and Vanhecke, 1979]. *Assume that $V_p(r) = V_0(r)$ for any $p \in M$ and small $r > 0$ i.e. $B_{2k} \equiv 0$ for any $k \in \mathbf{N}$. Then (M, g) is flat.*

In general case this conjecture is open. A. Gray and L. Vanhecke [GV] has proved that, under some assumptions on the dimension n and/or the curvature, the conjecture is true by calculating the first three invariants B_2, B_4, B_6 explicitly in terms of the curvature tensor, the Ricci tensor, the scalar curvature and their covariant derivative. Moreover, they constructed an example of a non-flat homogeneous Riemannian manifold such that $V_p(r) = V_0(r)(1 + O(r^8))$ for each $p \in M$ ([GV]). O. Kowalski [K] defined *the additive volume invariants* of a Riemannian manifold (cf. the next section) and proved the following theorem by using them.

THEOREM [K]. *There exists a non-flat homogeneous Riemannian manifold such that $V_p(r) = V_0(r)(1 + O(r^{16}))$ for each $p \in M$.*

This paper is concerned with the following question: *For given $k \in \mathbf{N}$, does there exist a non-flat homogeneous Riemannian manifold such that $V_p(r) = V_0(r)(1 + O(r^{2k+2}))$?* We will prove the following.

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