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Dual Class of a Subvariety

Tatsuo SUWA

Hokkaido University

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Let M be a complex manifold of dimension n and E a holomorphic vector bundle of rank k over M. If s is a regular section of E (cf. [F] B.3), it defines an analytic subspace X of pure codimension k in M. It is "well-known" that, if M is compact, then the top Chern class $c_k(E)$ of E corresponds to the homology class $[X]$ of X under the Poincaré duality $P : H^{2k}(M; \mathbf{C}) \xrightarrow{\sim} H_{2n-2k}(M; \mathbf{C})$ (in fact this holds with $\mathbf{Z}$ coefficients). The nature of the proof of this fact depends on how one defines the class $c_k(E)$ (cf. [G] §5 for the projective non-singular case, [F] §14.1 for the general case in the algebraic category and [GH] Ch. 1, §1 for the case $k = 1$ in the complex analytic category). In this article, we take up the definition of Chern classes via the Chern-Weil theory and give a relatively elementary proof of a more precise statement in the complex analytic category. Namely, we prove the following. Let V denote the support of X, then there is a canonical localization $c_k(E, s)$, in the relative cohomology $H^{2k}(M, M \setminus V; \mathbf{C})$, of $c_k(E)$ with respect to s and, if V is compact (M may not be), the class $c_k(E, s)$ corresponds to $[X]$ under the Alexander duality

$$A : H^{2k}(M, M \setminus V; \mathbf{C}) \xrightarrow{\sim} H_{2n-2k}(V; \mathbf{C})$$

(Theorem 4.2). If M is compact, we have the commutative diagram

$$\begin{array}{ccc} H^{2k}(M, M \setminus V; \mathbf{C}) & \xrightarrow{j^*} & H^{2k}(M; \mathbf{C}) \\ \wr \downarrow A & & \wr \downarrow P \\ H_{2n-2k}(V; \mathbf{C}) & \xrightarrow{i_*} & H_{2n-2k}(M; \mathbf{C}) \end{array}$$

where i and j denote the inclusions $V \hookrightarrow M$ and $(M, \emptyset) \hookrightarrow (M, M \setminus V)$, respectively. Since $j^*(c_k(E, s)) = c_k(E)$, we recover the result we first mentioned. For an application, see [S2].

As related topics, we discuss intersections of analytic subspaces. We also prove a duality theorem when V as above may not be compact, considering X as a relative cycle in M modulo $M \setminus S$ for a compact connected component S of its singular set (Theorem 6.4). This fact is effectively used in [BLSS]. The proofs of the above results are done in the framework of Čech-de Rham cohomology.

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