TOKYO J. MATH.
VOL. 23, NO. 2, 2000

# Character Sums Attached to Finite Reductive Groups

Naomichi SAITO and Ken-ichi SHINODA

*Sophia University*

## Introduction.

In [12], Kondo determined the value of Gaussian sum for every irreducible representation of $GL_n(q)$ and Macdonald also treated this problem in [14]. Recently in a series of papers, Kim-Lee [4], Kim ([5], [6], [7], [8], [9], [10], [11]), and Lee-Park [13] determined the values of Gaussian sums for one-dimensional representations of finite classical groups and $G_2(q)$.

In this note, we firstly show that a character sum over a finite reductive group associated with the generalized character $R_{\mathrm{T},\theta}$ of Deligne-Lusztig is reduced to a character sum over a torus. Applying this result to Gaussian sums and Kloosterman sums attached to finite classical groups, we obtain explicit formulae of these sums related with $R_{\mathrm{T},\theta}$, when $\pm R_{\mathrm{T},\theta}$ is irreducible. Also combining this result with the Davenport-Hasse type relations of Kloosterman sums and unitary Kloosterman sums proved in [2], we can explicitly determine the values of these sums for every irreducible character if the rank of the group is low. As an example, we give a table of Gaussian sums attached to $Sp_4(q)$, with $q$ odd. In Section 3, Kloosterman sums over $GL_n(q)$ are considered, and the properties and conjectures of these sums for unipotent characters are given.

The second named author would like to express his sincere gratitude to Masao Koike and Charles W. Curtis for their valuable discussions and comments.

NOTATION. We shall use the similar notation as in [2]. In particular $\mathbb{F}_q$ denotes a finite field with $q$ elements, and $\mathbb{F}_{q^m}$ the extension field of degree $m$ of $\mathbb{F}_q$, contained in a fixed algebraic closure $\bar{\mathbb{F}}_q$ of $\mathbb{F}_q$. $C_m = \{\alpha \in \mathbb{F}_{q^{2m}} : \alpha^{q^m+1} = 1\}$ is the cyclic group of order $q^m + 1$ in $\mathbb{F}_{q^{2m}}^\times$ and we will write $C = C_1$. If $m$ divides $n$, $\mathrm{Tr}_{\mathbb{F}_{q^n}/\mathbb{F}_{q^m}} : \mathbb{F}_{q^n} \to \mathbb{F}_{q^m}$ is the trace map. We fix a nontrivial additive character $\chi$ of $\mathbb{F}_q$ throughout this paper, and put $\chi^{(m)} = \chi \circ \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}$, the canonical lift of $\chi$ to $\mathbb{F}_{q^m}$. For a multiplicative character $\pi$ of $\mathbb{F}_q^\times$, the sum

$$K(\chi, \pi, a) = \sum_{st=a} \chi(s+t)\pi(s), \quad a \in \mathbb{F}_q^\times$$

Received February 4, 1999