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On the Special Functions Higher than the Multiple Gamma-Functions

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The present article can be viewed as a continuation of Katayama-Ohtsuki [5] and roughly speaking, concerns with the special functions derived from the multiple Riemann zeta-functions by differentiating twice. Such functions are necessary for studying zeta- or L-function whose functional equation involves $\Gamma(s)^r$, $r \geqq 2$, from the view point, at least, of the theory of Shintani L-functions. Also, for example, study of Kronecker's limit formula for Eisenstein series of Hilbert type, which will be a next task, demands our special functions since the series have $\Gamma(s)^r$, $r \geqq 2$, as Γ-factors in their functional equations.

Our theory will go quite parallel to [5] and its key is Lemma 1 in §2 which is analogous to Lemma 1 in [5]. In §1, first we quote necessary facts on multiple gamma function from Shintani [6] and we introduce new special functions: namely (r, k)-gamma functions ${}_k\Gamma_r(w; \tilde{\omega})$, Stirling (r, k)-modular forms ${}_k\rho_r(\tilde{\omega})$, $L\Gamma_r(w; \tilde{\omega})$ and auxiliary functions $L_2G_m(z)$. But we shall mainly concern with latter two. Their definitions are quite parallel to that of multiple gamma function and $LG(z)$ by Shintani [6], [7]. In §2, we derive asymptotic expansions of them on the basis of the key Lemma 1.

In §3, we construct the above functions by Weierstrass principle. In §4, the special cases for $r = 0, 1$, are considered for supplying our theory.

1. The definition of the function $L\Gamma_r(w; \tilde{\omega})$.

1.1. The multiple Riemann zeta-function. Let $w, \omega_1, \cdots, \omega_r$ be complex numbers with positive real parts. Then r-ple Riemann zeta-function ζ_r is defined by

$$(1.1.1) \qquad \zeta_r(s; w; \tilde{\omega}) = \sum_{\tilde{m}=\tilde{0}}^{\infty} (w + m_1\omega_1 + \cdots + m_r\omega_r)^{-s}, \quad \operatorname{Re} s > r,$$

where $\tilde{\omega} = (\omega_1, \cdots, \omega_r)$, $\tilde{0} = (0, \cdots, 0)$ and $\tilde{m} = (m_1, \cdots, m_r)$, $m_i \in \mathbf{Z}$, $m_i \geqq 0$. Here

$$w^s = \exp(s \log w),$$
$$\log w = \log |w| + i \arg w, \quad -\pi < \arg w < \pi.$$

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