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On Correspondences between Once Punctured Tori and Closed Tori: Fricke Groups and Real Lattices

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1. Introduction.

We consider the Teichmüller space of the closed torus and the Teichmüller space of the once punctured torus. It is well-known that the former can be identified with the upper half-plane and that several coordinate systems can be introduced to the latter. This is the first part of a series of papers in which we investigate explicit relations between these two Teichmüller spaces. In this paper based on a correspondence of subsets of these spaces we will give an explicit construction of a holomorphic mapping between a once punctured torus and a closed torus.

We use throughout the convention that an element A in $\mathrm{PSL}(2, \mathbf{R})$ represents the Möbius transformation induced by A, *i.e.*,

$$\text{if } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2, \mathbf{R}) \text{ then } A(z) = \frac{az+b}{cz+d}.$$

We consider a Fuchsian group G consisting of Möbius transformations of $\mathrm{PSL}(2, \mathbf{R})$ and having the following properties: (i) G is discontinuous in the upper half-plane $\mathbf{H}$, (ii) every real number is a limit point for G, (iii) G is finitely generated.

DEFINITION 1.1. A Fuchsian group $\Gamma = \langle A, B \rangle$ for $A, B \in \mathrm{PSL}(2, \mathbf{R})$ is called a *Fricke group* if A, B are hyperbolic and $\mathrm{tr}[B^{-1}, A^{-1}] = -2$.

In the definition above $\Gamma = \langle A, B \rangle$ is the free group generated by A, B and tr denotes the trace of a matrix. We consider a once punctured torus which is uniformized by a Fricke group Γ and take a normalized form for the presentation of Γ (see §5). By using the quantities $X = \mathrm{tr}\, A$, $Y = \mathrm{tr}\, B$ and $Z = \mathrm{tr}\, AB$, the above description of the Fricke group is characterized by $X^2 + Y^2 + Z^2 = XYZ$ and $X, Y, Z > 2$. Moreover, we obtain the following theorem (see [W]).

THEOREM 1.1 (Fricke [F], Keen [K]). *The Teichmüller space $\mathcal{T}_{1,1}$ of the once punctured torus is the sublocus of $X^2 + Y^2 + Z^2 = XYZ$ with $X, Y, Z > 2$.*

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