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On a Diophantine Equation Concerning Eisenstein Numbers

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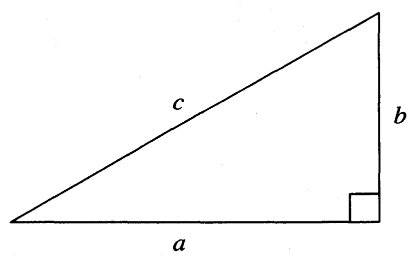
1. Introduction

In 1956, Jeśmanowicz [J] conjectured that if a, b, c are *Pythagorean numbers*, i.e., positive integers satisfying $a^2 + b^2 = c^2$, then the Diophantine equation

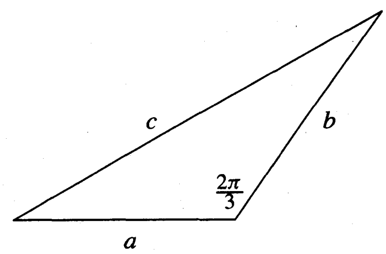
$$a^x + b^y = c^z$$

has only the positive integral solution $(x, y, z) = (2, 2, 2)$. It has been verified that this conjecture holds for many Pythagorean numbers (cf. Sierpiński [S1], [S2], [TA1], [TA2], [Ta1], [Ta2], [GL] and [Le]). This conjecture, however, is still open.

If a, b, c are positive integers satisfying $a^2 + ab + b^2 = c^2$, we call a, b, c *Eisenstein numbers*. Eisenstein numbers have some properties similar to those of Pythagorean numbers. As shown in Lemma 1 below, Eisenstein numbers $a,\ b,\ c$ can be expressed in terms of positive integers $u,\ v$ by factoring $a^2 + ab + b^2 = c^2$ in $\mathbf{Q}(\omega)$, where $\omega = e^{2\pi i/3} = (-1 + \sqrt{-3})/2$. It is worth noting that, geometrically, Pythagorean numbers a, b, c are the sides of a right triangle, and that Eisenstein numbers a, b, c are the sides of a triangle with an interior angle $2\pi/3$. See the figures below.



Pythagorean numbers a, b, c.



Eisenstein numbers a, b, c.

As an analogue to Jeśmanowicz' conjecture, we propose the following (cf. Terai [Te1], [Te2]):

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