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On Numerical Types of Algebraic Curves on Rational Surfaces

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1. Introduction.

The purpose of this paper is to study numerical properties of algebraic curves on rational surfaces. Here by curves and surfaces we mean projective irreducible varieties of dimension 1 and 2, respectively, which are defined over the field of complex numbers. We shall study pairs (S, D) of projective non-singular rational surfaces S and curves D on S.

First, we recall some basic notions and results concerning birational geometry of pairs. Let Σ_b denote a $\mathbf{P}^1$-bundle over $\mathbf{P}^1$ that has a minimal section Δ_∞ with $\Delta_\infty^2 = -b \leq 0$. Furthermore, let C be a curve on Σ_b. The Picard group of Σ_b is generated by the section Δ_∞ and a fiber $F_u = \rho^{-1}(u)$ of the $\mathbf{P}^1$-bundle Σ_b where ρ is the projection (cf. [H1, p. 370, Proposition 2.3]). Then $C \sim \sigma\Delta_\infty + eF_u$ for some integers σ and e. Here the symbol $\sim$ indicates linear equivalence between divisors. If $b \geq 1$, (σ, e) is uniquely determined. But when $b = 0$, the Σ_0 has two $\mathbf{P}^1$- bundle structures. In this case, we may assume that $e \geq \sigma$. Then (σ, e) is uniquely determined and we say that (Σ_b, C) has the *degree* (σ, e). Moreover, let m_1 denote the highest multiplicity of singular points of C. The pair (Σ_b, C) is said to be *#-minimal* if $\sigma \geq 2m_1$ and $e - \sigma \geq bm_1$ (cf. [I1]). The last condition is always satisfied whenever $b \geq 2$.

Let D be a non-singular curve on S. Then the pair (S, D) is said to be *relatively minimal*, whenever the intersection number $D \cdot E \geq 2$ for any exceptional curve E of the first kind on S such that $E \neq D$ (cf. [I1], Theorem 1, [S]). Suppose that (S, D) is a relatively minimal pair such that $S \neq \mathbf{P}^2$ and $\kappa[D] \geq 0$ where $\kappa[D]$ denotes $\kappa(K + D, S)$. Then there exists a #-minimal pair (Σ_b, C) such that (S, D) is derived from (Σ_b, C) by resolving singularities on C in a shortest way (cf. [I1]). In this case, we say that (Σ_b, C) is a *#-minimal model* of (S, D). The structure of (S, D) with $\kappa[D] \leq 1$ has been precisely determined by Iitaka in [I1] and [I2]. If $\kappa[D] = 2$, then relatively minimal pairs are always minimal (see [I1]), and hence, D^2 is invariant for birational transformation of pairs. Note that if $\kappa[D] \geq 0$, then σ

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