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Good Ideals in Artinian Gorenstein Local Rings Obtained by Idealization

Shiro GOTO and Satoshi HARAIKAWA

Meiji University

1. Introduction.

The purpose of this note is to prove the following, which gives a structure theorem of certain ideals in Artinian Gorenstein local rings obtained by idealization.

THEOREM 1.1. *Let R be an Artinian local ring with the maximal ideal $\mathfrak{n}$ and let $E = E_R(R/\mathfrak{n})$ denote the injective envelope of $R/\mathfrak{n}$. Let $A = R \ltimes E$ be the idealization of E over R and let I be an ideal in A. Then the following two conditions are equivalent.*

(1) $I = (0) : I$.

(2) *There exists a pair $(\mathfrak{a}, h)$, where $\mathfrak{a}$ is an ideal in R with $\mathfrak{a}^2 = (0)$ and $h : L := (0) :_E \mathfrak{a} \to R/\mathfrak{a}$ is a homomorphism of $R/\mathfrak{a}$-modules, satisfying the following four conditions*:

(a) $h(x)h(y) = 0$ *and* $h(x)y + h(y)x = 0$ *for all* $x, y \in L$.

(b) *Let $a, b \in R$. Then $ab = 0$ if $\bar{a}, \bar{b} \in h(L)$. (Here $\bar{*}$ denotes the reduction* mod $\mathfrak{a}$.)

(c) *Let $a \in R$ with $\bar{a} \in h(L)$. Then $ax \in L$ and $h(ax) = 0$ for all $x \in E$.*

(d) $I = \{(a, x) \mid a \in R,\ x \in L$ *such that* $\bar{a} = h(x)\}$.

When this is the case, the pair $(\mathfrak{a}, h)$ is uniquely determined by I and $\mathfrak{a} = f^{-1}(I)$, where $f : R \to A$, $f(a) = (a, 0)$ denotes the structure map.

Let A be a Gorenstein local ring with the maximal ideal $\mathfrak{m}$ and let I be an $\mathfrak{m}$-primary ideal in A. Then, following [GIW], we say that I is a *good* ideal in A, if I contains a parameter ideal Q in A as a reduction and the associated graded ring $\mathrm{G}(I) = \bigoplus_{n \geq 0} I^n/I^{n+1}$ of I is a Gorenstein ring with $\mathrm{a}(\mathrm{G}(I)) = 1 - \dim A$, where $\mathrm{a}(\mathrm{G}(I))$ denotes the a-invariant of $\mathrm{G}(I)$ ([GW, Definition 3.1.4]). This is a condition equivalent to saying that $I^2 = QI$ and $I = Q : I$, that is $I^2 = QI$ and the equality $\ell_A(A/I) = \frac{1}{2}\ell_A(A/Q)$ holds true ([GIW, Propositon 2.2]), where $\ell_A(*)$ stands for the length. Therefore, the first condition (1) in Theorem 1.1 is just equivalent to saying that I is a good ideal in $A = R \ltimes E$. In [GIW] the first author, Iai, and Watanabe intensively studied general Gorenstein local rings of arbitrary dimension and established many interesting results on good ideals. Nevertheless, in our very special situation

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