TOKYO J. MATH.
VOL. 25, NO. 2, 2002

# Examples of Compact Lefschetz Solvmanifolds

Takumi YAMADA

*Osaka University*

(Communicated by R. Miyaoka)

## Introduction.

Let $(M^{2m}, \omega)$ be a compact symplectic manifold. A symplectic manifold $(M, \omega)$ is called a Lefschetz manifold if the mapping $\wedge\omega^{m-1} : H^1_{DR}(M) \to H^{2m-1}_{DR}(M)$ is an isomorphism. We also say that $(M, \omega)$ has the Hard Lefschetz property, if the mapping $\wedge\omega^k$ : $H^{m-k}_{DR}(M) \to H^{m+k}_{DR}(M)$ is an isomorphism for each $k \leq m$. By a solvmanifold we mean a homogeneous space $G/\Gamma$, where $G$ is a simply-connected solvable Lie group and $\Gamma$ is a lattice, that is, a discrete co-compact subgroup of $G$. A solvable Lie algebra $\mathfrak{g}$ is called completely solvable if $\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ has only real eigenvalues for each $X \in \mathfrak{g}$. Benson and Gordon [BG1] have proved that no non-toral compact nilmanifolds are Lefschetz manifolds for any symplectic structure to show that a non-toral compact nilmanifold does not admit any Kähler structure. Moreover, they conjecture the following :

BENSON-GORDON CONJECTURE [BG2]. *Let $G$ be a simply-connected completely solvable Lie group and $\Gamma$ a lattice of $G$. Then a compact solvmanifold $G/\Gamma$ admits a Kähler structure if and only if it is a torus.*

The authors of [AFLM] and [FLS] have constructed examples of 6-dimensional compact Lefschetz solvmanifolds with the Hard Lefschetz property and without the Hard Lefschetz property (See Example 5.1 and 5.4). More precisely, let $G_6$ be the simply-connected completely solvable Lie group defined by

$$G_6 = \left\{ \begin{pmatrix} e^t & 0 & xe^t & 0 & 0 & 0 & y_1 \\ 0 & e^{-t} & 0 & xe^{-t} & 0 & 0 & y_2 \\ 0 & 0 & e^t & 0 & 0 & 0 & y_3 \\ 0 & 0 & 0 & e^{-t} & 0 & 0 & y_4 \\ 0 & 0 & 0 & 0 & 1 & 0 & x \\ 0 & 0 & 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \middle| \; t, x, y_1, y_2, y_3, y_4 \in \mathbf{R} \right\} .$$

Received July 3, 2001; revised January 28, 2002