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On Two Step Tensor Modules of the Maximal Compact Subgroups of Inner Type Noncompact Real Simple Lie Groups

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1. Introduction

Let $\mathbf{C}$ (resp. $\mathbf{R}$) be the complex (resp. real) number field. We consider a connected simply connected complex simple Lie group $G_{\mathbf{C}}$ and its connected noncompact simple real form G. In this article we shall always fix a maximal compact subgroup K of G, and assume that rank G = rank K. This assumption is equivalent to G is inner. Let $\mathfrak{g}$ and $\mathfrak{k}$ be respectively the Lie algebras of G and K. Let θ be the Cartan involution which stabilizes K. Then $\mathfrak{g}$ is decompsed by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{p}$ is the eigenspace of θ in $\mathfrak{g}$ with the eigenvalue -1. Let $\mathfrak{g}_{\mathbf{C}}$ be the Lie algebra of $G_{\mathbf{C}}$. We shall denote, for each subspace $\mathfrak{v}$ of $\mathfrak{g}$, by $\mathfrak{v}_{\mathbf{C}}$ the complexification of $\mathfrak{v}$ in $\mathfrak{g}_{\mathbf{C}}$. $\mathfrak{p}_{\mathbf{C}}$ is a K-module by the adjoint action of K. Let B be a maximal abelian subgroup of K. Then B is also a maximal abelian subgroup (Cartan subgroup) of G. Let $\mathfrak{b}$ be the Lie algebra of B. Then the root system Σ of the pair $(\mathfrak{g}_{\mathbf{C}}, \mathfrak{b}_{\mathbf{C}})$ is decomposed by $\Sigma = \Sigma_K \cup \Sigma_n$, where Σ_K (resp. Σ_n) is the set of all compact (resp. noncompact) roots in Σ. Then Σ_K is also the root system of $(\mathfrak{k}_{\mathbf{C}}, \mathfrak{b}_{\mathbf{C}})$. We choose a positive root system P_K, and always fix it.

Let us state our purpose of this article. Let μ be a P_K-dominant integral form on $\mathfrak{b}_{\mathbf{C}}$ and (π_μ, V_μ) a simple K-module with highest weight μ. We consider a simple Harish-Chandra $(\mathfrak{g}, K)$-module $U(\mathfrak{g}_{\mathbf{C}})V_\mu$ which contains (π_μ, V_μ) with multiplicity one, where $U(\mathfrak{g}_{\mathbf{C}})$ is the universal enveloping algebra of $\mathfrak{g}_{\mathbf{C}}$. Let $\mathfrak{p}_{\mathbf{C}} \otimes \mathfrak{p}_{\mathbf{C}} \otimes V_\mu$ be the tensor K-module. Canonically this space has a unitary K-module structure. We define a K-linear homomorphism ϖ of $\mathfrak{p}_{\mathbf{C}} \otimes \mathfrak{p}_{\mathbf{C}} \otimes V_\mu$ to $U(\mathfrak{g}_{\mathbf{C}})V_\mu$ by $\varpi(X \otimes Y \otimes v) = XYv$ for $X, Y \in \mathfrak{p}_{\mathbf{C}}$, $v \in V_\mu$. Let V be a finite K-module. We define a projection operator P_μ on V by

$$P_\mu(v) = \deg \pi_\mu \int_K kv\,\overline{trace\pi_\mu(k)}dk \quad \text{for } v \in V\,, \tag{1.1}$$

where $\deg \pi_\mu = \dim V_\mu$ and dk is the Haar measure on K normalized as $\int_K dk = 1$. Since $P_\mu \varpi = \varpi P_\mu$, ϖ induces a K-module linear homomorphism of $M(\mu) = P_\mu(\mathfrak{p}_{\mathbf{C}} \otimes \mathfrak{p}_{\mathbf{C}} \otimes V_\mu)$ to $V_\mu \subset U(\mathfrak{g}_{\mathbf{C}})V_\mu$. Let $m = m(\mu)$ be the multiplicity of V_μ in $M(\mu)$. $M(\mu)$ is decomposed by $M(\mu) = \bigoplus_{j=1}^m U(\mathfrak{k}_{\mathbf{C}})v_j$, where v_j is the highest weight vector of the simple K-module $U(\mathfrak{k}_{\mathbf{C}})v_j$ and $U(\mathfrak{k}_{\mathbf{C}})$ is the universal enveloping algebra of $\mathfrak{k}_{\mathbf{C}}$. Let $v(\mu)$ be the highest weight

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