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The Gonality of Singular Plane Curves

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1. Introduction

Let $C \subset \mathbf{P}^2$ be an irreducible plane curve of degree d over the complex number field $\mathbf{C}$. We denote by $\mathbf{C}(C)$ the field of rational functions on C. Let $\tilde{C}$ be the non-singular model of C. Since $\mathbf{C}(\tilde{C}) \cong \mathbf{C}(C)$, a non-constant rational function φ on C induces a non-constant morphism $\varphi : \tilde{C} \to \mathbf{P}^1$. Let deg φ denote the degree of this morphism φ. We remark that deg $\varphi = [\mathbf{C}(C) : \mathbf{C}(\varphi)] = \deg(\varphi)_0 = \deg(\varphi)_\infty$. The *gonality* of C, denoted by $\mathrm{Gon}(C)$, is defined to be $\min\{\deg \varphi \mid \varphi \in \mathbf{C}(C) \setminus \mathbf{C}\}$. So by definition, the gonality of C is nothing but the gonality of $\tilde{C}$. Let ν denote the maximal multiplicity of C. We easily see that $\mathrm{Gon}(C) \leq d - \nu$. We know that the genus of C is equal to $(d-1)(d-2)/2 - \delta$ with $\delta \geq 0$.

THEOREM 1. *Let C be an irreducible plane curve of degree d with $\delta \geq \nu$. Letting $d \equiv i \pmod{\nu}$, define*

$$R(\nu, \delta, i) = \frac{\nu^2 + (\nu - 2)i}{2\nu(\nu - 1)} + \sqrt{\frac{\delta - \nu}{\nu - 1} + \left(\frac{\nu - 2 + i}{2(\nu - 1)}\right)^2}.$$

If $d/\nu > R(\nu, \delta, i)$, then $\mathrm{Gon}(C) = d - \nu$.

REMARK 1. Theorem 1 is a generalization of Theorem 2.1 in Coppens and Kato [1] where they considered the case in which C has only nodes and ordinary cusps. Note that $R(2, \delta, 0) = 1 + \sqrt{\delta - 2}$, $R(2, \delta, 1) = 1 + \sqrt{\delta - 7/4}$. In general, we have the estimation: $R(\nu, \delta, i) < 1 + \sqrt{\delta/(\nu - 1)}$.

We have $\delta < \nu$ if either (i) C is a smooth curve ($\delta = 0$, $\nu = 1$ and $\mathrm{Gon}(C) = d - 1$ for all $d \geq 2$), or (ii) C has one node or one ordinary cusp ($\delta = 1$ and $\nu = 2$ and $\mathrm{Gon}(C) = d - 2$ for all $d \geq 3$). Cf. [1], [3], [5].

DEFINITION. Let $m_1, \cdots, m_n$ denote the multiplicities of all singular points (we include infinitely near singular points) of C. Set $\eta = \sum (m_i/\nu)^2$. Clearly, we have $n \geq \eta \geq 1$.

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