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Barnes' Multiple Zeta Function and Apostol's Generalized Dedekind Sum

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0. Introduction

The present paper can be viewed as a continuation of [6], where we treated the case of Barnes' double zeta function. Here we handle mainly the case of Barnes' triple zeta function and prove the three term formula for odd Apostol's generalized Dedekind sum. It seems to be new.

Our method is the same as [6], namely we compute the contour integral representation of the Barnes' triple zeta function. Section 1 concerns the general Barnes multiple and twisted multiple Bernoulli polynomials. This, in section 2, enables us to get generally residues at poles of the integrand of the contour integral of the Barnes multiple zeta function. In section 3, we shall quote results of Apostol[1] for later use and derive some formulas relative to Lambert series and Apostol's generalized Dedekind sum.

In the last of section 3, we derive the formula to be called the "three term formula" for odd Apostol's generalized Dedekind sum.

1. Multiple Bernoulli numbers and polynomials twisted by $\tilde{\alpha}$

1.1. Barnes' polynomial. Barnes [2] (cf. [7]) introduced r-ple Bernoulli polynomials ${}_rS_n(u;\tilde{\omega})$ by

$$\frac{(-1)^r t e^{-ut}}{\prod_{i=1}^r (1-e^{-\omega_i t})} = \sum_{k=1}^{r} \frac{(-1)^k {}_rS_1^{(k+1)}(u;\tilde{\omega})}{t^{k-1}} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} {}_rS_n'(u;\tilde{\omega})}{n!} t^n \tag{1.1.1}$$

for $|t| < \min\{|2\pi/\omega_1|, \cdots, |2\pi/\omega_r|\}$. Here $\omega_1, \cdots, \omega_r$ are complex numbers with positive real parts and $\tilde{\omega} = (\omega_1, \cdots, \omega_r)$. ${}_rS_1^{(k)}(u;\tilde{\omega})$ means the k-th derivative of ${}_rS_1(u;\tilde{\omega})$ with respect to u.

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