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On a Theorem of Kawamoto on Normal Bases of Rings of Integers

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1. Introduction

A finite Galois extension L/K over a number field K has a relative normal integral basis (NIB for short) when $\mathcal{O}_L$ is free over the group ring $\mathcal{O}_K[\mathrm{Gal}(L/K)]$. Here, $\mathcal{O}_L$ (resp. $\mathcal{O}_K$) is the ring of integers of L (resp. K). It is well known by Noether that if L/K has a NIB, then L/K is tame (i.e., at most tamely ramified at all finite prime divisors). It is also well known by Hilbert and Speiser that when the base field K equals the rationals $\mathbf{Q}$, all tame abelian extensions $L/\mathbf{Q}$ have a NIB. Recently, Greither et al. [3] proved that there exists no Hilbert-Speiser number field other than $\mathbf{Q}$. Namely, they proved that when $K \neq \mathbf{Q}$, there exist a prime number p and a tame cyclic extension L/K of degree p having no NIB.

On the other hand, Kawamoto [7, 8] obtained the following result. For a prime number p, let ζ_p be a fixed primitive p-th root of unity.

THEOREM 1 (Kawamoto). *For a prime number p and a rational number $a \in \mathbf{Q}^\times$, the cyclic extension $\mathbf{Q}(\zeta_p, a^{1/p})/\mathbf{Q}(\zeta_p)$ has a NIB if it is tame.*

In [2, Theorem 2.1], Gómez Ayala gave a necessary and sufficient condition for a tame Kummer extension of prime degree to have a NIB, and deduced Theorem 1 from this criterion. For a prime number p, we say that a number field F enjoys the property (H_p) when for any element $a \in F^\times$, the cyclic extension $F(\zeta_p, a^{1/p})/F(\zeta_p)$ has a NIB if it is tame. Theorem 1 says that the rationals $\mathbf{Q}$ satisfies the property (H_p) for all p. Analogous to the result of Greither et al., it is shown in [5, IV] that when $F \neq \mathbf{Q}$, there exists a prime number p for which F does not satisfy (H_p). For a prime number p and a number field F with $\zeta_p \in F^\times$, we gave, in [5, V, Propositions 1, 2], a necessary and sufficient condition for (H_p) to be satisfied.

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