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On Totally Real Cubic Orders Whose Unit Groups are of Type $\langle a\theta + b, c\theta + d\rangle$

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1. Introduction

Let $\phi(x)$ be a cubic, monic and irreducible polynomial in x with rational integer coefficients and three real roots. We fix one of these roots and denote it by θ. Set $K = \mathbf{Q}(\theta)$, and let E_K be the unit group of K and E_K^+ the subgroup of E_K which consists of units with norm $+1$. By Dirichlet's unit theorem, E_K^+ is generated by two units and so is $\mathbf{Z}[\theta] \cap E_K^+$. Hereafter we denote the latter by E_θ^+. It is difficult to determine the generators of E_θ^+ even though that problem is important for number theory. In this paper, for given $a, b, c, d \in \mathbf{Z}$, we shall find conditions under which $E_\theta^+ = \langle a\theta + b, c\theta + d\rangle$. As a result, we shall obtain new infinite families of $\mathbf{Z}[\theta]$ with explicit generators of E_θ^+, which will give useful examples for further study.

In 1972, Stender[6] found families of $\phi(x)$ such that $E_\theta^+ = \langle \theta + b, \theta + d\rangle$ for rational integers b, d with $2 \le b \le d - 3$ by using Berwick's theorem[1]. In 1979, Thomas [7] found families of $\phi(x)$ such that $E_\theta^+ = \langle a\theta + 1, \theta + d\rangle$ and $\langle a\theta + 1, c\theta + 1\rangle$ for rational integers a, c, d with $a \ge 4$ and some other conditions by using the continued fraction expansion of a certain conjugate of θ. In 1995, Grundman [3] modified Thomas's technique for determining fundamental systems of units, and determined all a with $|a| > 1$ such that $E_\theta^+ = \langle a\theta + 1, 2\theta + 3\rangle$ for some totally real number θ of degree 3, and found families of $\phi(x)$ for each a. We shall further utilize this method under a more general condition that $a\theta + b, c\theta + d \in E_\theta^+$.

THEOREM 1. *For rational integers a, b, c and d, assume the following conditions*:
1. $|ad - bc| > \max\{|a|, |c|\}, 2 \le |a| < |b|$ *and* $2 \le |c| < |d|$,
2. *there exist rational integers e, f and g such that*

$$b^3 - eab^2 + fa^2b - ga^3 = 1\,, \quad d^3 - ecd^2 + fc^2d - gc^3 = 1\,, \qquad (1)$$

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