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On a Higher Class Number Formula of $\mathbf{Z}_p$-Extensions

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1. Introduction

Let p be a prime number and k a number field of finite degree over $\mathbf{Q}$, the rational number field. Let $\mathbf{Z}_p$ be the additive group of p-adic integers, and K/k a $\mathbf{Z}_p$-extension over k. For an integer $n \geq 0$, we denote by k_n the n-th layer of the extension K/k, namely k_n is the unique intermediate field of K/k such that $[k_n : k] = p^n$. Recently, Ozaki studied the maximal unramified pro-p extensions $\tilde{L}$ of K and $\tilde{L}_n$ of k_n as in what follows. Let $\tilde{G} = \mathrm{Gal}(\tilde{L}/K)$ and $\tilde{G}_n = \mathrm{Gal}(\tilde{L}_n/k_n)$ for all non-negative n. We define the subgroups $C_i(\tilde{G})$ of $\tilde{G}$ by the descending central series

$$\tilde{G} = C_1(\tilde{G}) \supseteq C_2(\tilde{G}) \supseteq \cdots \supseteq C_i(\tilde{G}) \supseteq \cdots, \quad C_{i+1}(\tilde{G}) = \overline{[C_i(\tilde{G}), \tilde{G}]}\,.$$

Then we consider the modules $X^{(i)} = C_i(\tilde{G})/C_{i+1}(\tilde{G})$, and call $X^{(i)}$ the i-th Iwasawa module. We define the subgroups $C_i(\tilde{G}_n) \subseteq \tilde{G}_n$ and the modules $X_n^{(i)}$ similar to $C_i(\tilde{G})$ and $X^{(i)}$, respectively. Note that $X_n^{(1)}$ is isomorphic to the Sylow p-subgroup of the ideal class group A_{k_n} of k_n and that $X^{(1)}$ is the Iwasawa module X_K of K/k which is defined as the projective limit $\varprojlim A_{k_n}$ with respect to the norm maps. By definition, the complete group ring $\Lambda_{K/k} = \mathbf{Z}_p[[\mathrm{Gal}(K/k)]]$ acts on $X^{(i)}$ in the natural way, namely $\mathrm{Gal}(K/k)$ acts via the inner automorphism. For $i = 1$, Iwasawa studied the $\Lambda_{K/k}$-module structure of X_K and deduced the following celebrated formula:

THEOREM A. *There exist non-negative integers $\lambda(K/k)$, $\mu(K/k)$ and an integer $\nu(K/k)$ such that*

$$\# A_{k_n} = p^{\lambda(K/k)n + \mu(K/k)p^n + \nu(K/k)}$$

for all sufficiently large n.

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