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A Remark on the Breuer's Conjecture Related to the Maillet's Matrix

Izumi KURIBAYASHI

Hamamatsu University

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Introduction

We are concerned with a question asked by F. Momose whether the $\phi(n)/2$ real numbers $\cot(u\pi/n)$ $(0 < u < n/2, (u, n) = 1)$ are independent over $\mathbf{Q}$ or not. Here $\phi(n)$ denotes Euler's phi function. In his book [1], T. Breuer settled the equivalent problem (cf. Proposition 1.3 and Proposition 3.2 below) in the case where n is a prime power with $n > 2$, and conjectured the $\mathbf{Q}$-independence for any n with $n > 2$ (cf. Conjecture C.12 in [1]). In this note, reducing to the nonvanishing of $L(1, \chi)$, we prove the following, which implies the $\mathbf{Q}$-independence of $\cot(u\pi/n)$'s (cf. §2).

PROPOSITION. *Let n be an integer with $n > 2$. Then the rank of the matrix* $\left(\frac{1}{2} - \left\langle \frac{au_b^*}{n} \right\rangle\right)$ *is equal to $\phi(n)/2$, where a, b range over the set $\{1, \ldots, n-1\}$ and $\{1, \ldots, \phi(n)/2\}$ respectively.*

As for notations such as $\langle\ \rangle$, u_b, u_b^*, see Notation below.

The Maillet's determinant, that is the determinant of the matrix in Proposition above in the case where a ranges over the set $\{u_1, \ldots, u_{\phi(n)/2}\}$, has been studied in various way (e.g., [2], [7], [6]). It is equal to the first factor of the class number, up to non-zero factor in the case where n is a prime power (cf. §3).

In this note, we deduce this Proposition by proving the following

THEOREM. *Let n be an integer with $n > 2$. Then the following holds*:

$$\begin{aligned}&\Delta(\zeta_n^{u_1}/(1-\zeta_n^{u_1}), \ldots, \zeta_n^{u_{\phi(n)/2}}/(1-\zeta_n^{u_{\phi(n)/2}}), 1, \zeta_n, \zeta_n^2, \ldots, \zeta_n^{\phi(n)/2-1})\\ &\quad = \pm\frac{2}{Qw}(n\sqrt{-1})^{\phi(n)/2} \cdot h_n^- \cdot d \cdot \prod_{\chi:odd} L_\chi ,\end{aligned}$$

where as for the notation $\Delta(t_1, \ldots, t_{\phi(n)})$ see Notation below and the other symbols denote

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