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# Some Problems on $p$-class Field Towers

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## 1. Introduction

Let $p$ be a prime number, $k$ a finite extension of the field of rational numbers $\mathbf{Q}$ and $L(k)$ the maximal unramified abelian $p$-extension of $k$. Let $\tilde{L}(k)$ be the maximal unramified pro-$p$ extension of $k$, which is called the $p$-class field tower of $k$, and denote its Galois group $\mathrm{Gal}(\tilde{L}(k)/k)$ over $k$ by $\tilde{G}_k$. Then $\mathrm{Gal}(L(k)/k) = \tilde{G}_k^{\mathrm{ab}} \simeq A_k$, the $p$-primary part of the ideal class group of $k$, by the class field theory. For a number field $k$ which can be an infinite extension of $\mathbf{Q}$, we use the same notation, e.g. $\tilde{L}(k)$ and $\tilde{G}_k$, as in a finite extension. Let $k_\infty$ be the cyclotomic $\mathbf{Z}_p$-extension over a number field $k$; in other words, $k_\infty$ is the unique infinite Galois subextension of the field obtained by adjoining to $k$ all roots of unity of $p$-power order, whose Galois group is isomorphic to the additive group of the ring $\mathbf{Z}_p$ of $p$-adic integers. Denote by $k_n$ the unique subextension of $k_\infty$ over $k$ of degree $p^n$.

Iwasawa theory of $\mathbf{Z}_p$-extensions deals with Galois groups of various abelian $p$-extensions over $k_n$ and $k_\infty$, in particular, of the maximal unramified abelian pro-$p$ extensions over $k_n$ and $k_\infty$. Recently, a number of mathematicians have been engaged in the study of non-abelian extensions of $k_\infty$ and $k_n$ using Iwasawa theory, especially the study of maximal pro-$p$ extensions with restricted ramification. For example, Ozaki [18] studied the maximal unramified pro-$p$ extension of $k_n$ for all $n$, and proved a non-abelian analogy of the Iwasawa class number formula. And from the point of view of the analogy between the theory of $\mathbf{Z}_p$-extensions of algebraic number fields and the theory of the Jacobian variety of algebraic curves, A. Schmidt and K. Wingberg study the Galois group of "the maximal positively ramified extensions over algebraic number fields", which is the analogy of the fundamental group of compact Riemann surfaces.

In such studies, the question which asks what kind of groups can appear as the Galois group $\tilde{G}_{k_\infty}$ of the maximal *unramified* pro-$p$ extension $\tilde{L}(k_\infty)$ of $k_\infty$, and what kind of properties characterize $\tilde{G}_{k_\infty}$ is an interesting problem for reasons of being concerned with the

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