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Partial Sums of Multiple Zeta Value Series

Chizuru SEKINE

Minami-katsushika High School

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1. Introduction

For $\vec{s} = (s_1, \ldots, s_d) \in \mathbf{N}^d$ with $s_d > 1$, the Multiplze Zeta Value (MZV) series $\zeta_d(\vec{s})$ is defined as

$$\zeta_d(\vec{s}) = \zeta_d(s_1, \ldots, s_d) = \sum_{1 \le k_1 < \cdots < k_d} k_1^{-s_1} \cdots k_d^{-s_d} .$$

We call d its depth and $wt(\vec{s}) := \sum_{i=1}^{d} s_i$ its weight.

Now we consider partial sums of these MZV series. More precisely, for $\vec{s} \in \mathbf{N}^d$ and a non-negative integer n, the nth partial sum of MZV series $H_d(\vec{s}; n)$ is defined by

$$H_d(\vec{s}\,; n) = H_d(s_1, \ldots, s_d\,; n) = \sum_{1 \le k_1 < \cdots < k_d \le n} k_1^{-s_1} \cdots k_d^{-s_d} , \tag{1.1}$$

where $H_d(\vec{s}\,; r) = 0$ for $r = 0, \ldots, d-1$. Then the following theorem was obtained by Wolstenholme ([3], p. 89):

THEOREM 1.1. *For any prime number* $p \ge 5$, $H_1(1; p-1) \equiv 0 \pmod{p^2}$.

In [6] Zhao studied the p-divisibility of $H_d(\vec{s}\,; p-1)$ for general $\vec{s}$, which turned out to be closely related to the Bernoulli numbers B_t defined by $\frac{x}{e^x-1} = \sum_{t=0}^{\infty} \frac{B_t}{t!} x^t$. When (p, t) is an irregular pair, i.e., $p \,|\, B_t$ for even t with $2 \le t \le p-3$, $H_d(\vec{s}\,; p-1)$ is divisible by higher power of p than usually expected. As another generalization of Theorem 1.1, Bayat considered the p-divisibility of $H_1^*(s\,; p^a - 1)$ for a positive integer a in [1], where $H_1^*(s\,; n) = \sum_{\substack{1 \le k \le n \\ p \nmid k}} k^{-s}$ for any $n \in \mathbf{N}$ and we have $H_1^*(s\,; p-1) = H_1(s\,; p-1)$.

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