TOKYO J. MATH.
VOL. 29, NO. 2, 2006

Volumes and Degeneration of Cone-structures on the Figure-eight Knot

Alexander MEDNYKH and Alexey RASSKAZOV

Sobolev Institute of Mathematics

(Communicated by M. Kurihara)

1. Introduction

A geometrical construction of manifolds on the figure-eight knot appeared originally in Thurston's lectures [22]. He constructed a hyperbolic three-dimensional manifold by gluing faces of two ideal tetrahedra. The manifold obtained in this way is homeomorphic to the complement of the figure-eight knot in the three-dimensional sphere $\mathbf{S}^3$. In addition, this manifold has a complete hyperbolic structure.

We define the *three-dimensional Euclidean cone-manifold* [11] to be the complete metric space obtained as the quotient of (possibly non-compact) geodesic 3-simplices in the three-dimensional Euclidean space $\mathbf{E}^3$ by an isometric gluing of faces in such a fashion that the underlying topological space is a manifold. In this case, the metric structure around each edge is defined by the cone angle, which is equal to the sum of the dihedral angles corresponding to the identified edges. We define the *singular set* of the cone-manifold to be the closure of the set of edges whose cone angle is not equal to 2π. By definition, on the complement of the singular set, the constructed space has a Euclidean structure. Analogously, we define *spherical* and *hyperbolic* cone-manifolds. A cone-manifold is said to be an *orbifold* if the cone angles are equal to $2\pi/n$ for an integer $n \in \mathbf{N}$.

Let $\mathcal{C}(2\pi/n)$ be an orbifold whose singular set forms a figure-eight knot, and whose cyclic isotropy group has order $n \geq 1$. It is well known [3, 9, 22], that the orbifold $\mathcal{C}(2\pi/n)$ is spherical for $n = 2$, Euclidean for $n = 3$ and hyperbolic for $n \geq 4$. In [8] Hilden, Lozano and Montesinos constructed a family of three dimensional cone-manifolds $\mathcal{C}(\theta)$ whose underlying space is the three-dimensional sphere and whose singular set is the figure-eight knot. They showed that the cone-manifold obtained is hyperbolic for $\theta \in [0, 2\pi/3)$, Euclidean for $\theta = 2\pi/3$ and spherical for $\theta \in (2\pi/3, \pi]$. They also calculated geometrical parameters for the fundamental polyhedra and volume formulas for complicated cone manifolds. The question of the existence of spherical structure for $\theta > \pi$ was left open.

Received September 1, 2003; revised August 29, 2006
This work has been supported by grant RFBR 03-01-00104 and grant INTAS 03-51-3663.