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Reducible Curves on Rational Surfaces

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1. Introduction

Let k be an algebraically closed field of characteristic zero. Throughout the present article we fix k as the ground field. Let X be a smooth projective surface and B a reduced curve on X. Then we define the m-genus $P_m[B]$ and the Kodaira dimension $\kappa[B]$ as follows (cf. [4]): Let $f : V \to X$ be a birational morphism such that the strict transform $D = f'(B)$ becomes a disjoint union of smooth curves. Then $P_m[B] := h^0(V, m(D + K_V))$ and $\kappa[B] := \kappa(D + K_V, V)$, where K_V is the canonical divisor on V and $\kappa(D + K_V, V)$ is the $(D + K_V)$-dimension of V (cf. [1]). Note that $P_m[B] = \bar{P}_m(V - D)$ and $\kappa[B] = \bar{\kappa}(V - D)$, where $\bar{P}_m(V - D)$ (resp. $\bar{\kappa}(V - D)$) denotes the logarithmic m-genus (resp. the logarithmic Kodaira dimension) of $V - D$ (cf. [1] and [11]).

Pairs (X, B) of smooth projective rational surfaces X and irreducible curves B on X were studied from the viewpoint of birational geometry by Iitaka [2, 5], Matsuda [9] and the others. In [3] and [4], Iitaka studied reducible curves B on smooth projective rational surfaces such that $\#(B) = 2$, where $\#(B)$ is the number of irreducible components of B. In particular, he proved the following result.

THEOREM 1.1. ([4]) *Let B be a reduced curve on a smooth projective rational surface X with $\#(B) \le 2$. Then $\kappa[B] = -\infty$ if and only if $P_2[B] = 0$.*

In the present article, by using the theory of open algebraic surfaces, we study reduced curves on smooth projective rational surfaces. In §3, we give a simple proof of Theorem 1.1 by using the structure theorems of open algebraic surfaces (cf. §2). In §4, we study the case where B consists of two rational curves and $\kappa[B] = 0$ or 1 and give a structure theorem of such a pair, which improves [4, Proposition 3]. In §5, we consider the relation of $P_m[B]$ and $\kappa[B]$ when $\#(B) \le 4$.

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