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A New Identity Relating Mock Theta Functions with Distinct Orders

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1. Introduction

S. Ramanujan listed 17 mock theta functions of orders 3, 5 and 7 in his last letter to G. H. Hardy. A mock theta function is a function $f(q)$ for $|q| < 1$ satisfying the following two conditions:

(i) For every root of unity ζ, there is a theta function $\theta_\zeta(q)$ such that the difference $f(q) - \theta_\zeta(q)$ is bounded as $q \to \zeta$ radially.

(ii) There is no single theta function which works as in (i) for all ζ: i.e., for every theta function $\theta(q)$ there is some root of unity ζ for which $f(q) - \theta(q)$ is unbounded as $q \to \zeta$ radially.

G. N. Watson found in [5] three more mock theta functions of order 3. Ramanujan gave more mock theta functions in his lost notebook. B. Gordon and R. J. McIntosh also found in [4] eight mock theta functions of order 8. A formal definition of order is unknown until now. It is, however, known that mock theta functions with the same order are related to each other except for order 7. In this paper we show an interesting new relation between mock theta functions with distinct orders in Theorem 1, and we further prove two new series representations of some 8th order mock theta functions in Theorem 2.

In section 2, we give the relation mentioned above in Theorem 1, which connects two mock theta functions with distinct orders, 3rd and 6th , to a generalized Lambert series.

In section 3, we introduce a function $F(q,t)$ defined by G. E. Andrews in [1]. There he consider three specializations of t; here we add to them the fourth specialization of t, and we will further show that there are relations, called half-shift in [4], among these four functions (see Definition 4 and below). We next give three examples of the function $F(q,t)$ in Propositions 1-3. The last one is particularly interesting, because we again see two mock theta functions appeared in section 2. Finally we give new series representations of two 8th order mock theta functions.

We close this section by introducing some notation.

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