The fundamental group of the moduli space of polygons in the Euclidean plane

By Shuichi Tsukuda

Department of Mathematical Sciences, University of the Ryukyus, Nishihara-cho, Okinawa 903-0213

(Communicated by Shigefumi Mori, m.j.a., June 15, 1999)

1. Introduction. Let $\mathcal{C}_n$ be the configuration space of planar polygons with n vertices, each edge having length 1 in $\mathbf{R}^2$;

$$\mathcal{C}_n = \{(u_1,\ldots,u_n) \mid |u_{i+1} - u_i| = 1 \ (1 \le i \le n-1),$$
$$|u_1 - u_n| = 1\} \subset (\mathbf{R}^2)^n.$$

Note that $\mathrm{Iso}(\mathbf{R}^2)$, the isometry group of $\mathbf{R}^2$, naturally acts on $\mathcal{C}_n$. We define

$$M_n = \mathcal{C}_n / \mathrm{Iso}^+(\mathbf{R}^2)$$
$$M_n' = \mathcal{C}_n / \mathrm{Iso}(\mathbf{R}^2)$$

where $\mathrm{Iso}^+(\mathbf{R}^2)$ is the orientation preserving isometry group. Identifying $\mathbf{R}^2$ with $\mathbf{C}$, we can write M_n as

$$M_n = \{(z_1,\ldots,z_{n-1}) \mid z_1 + \ldots + z_{n-1} - 1 = 0\}$$
$$\subset (S^1)^{n-1}$$

and $M_n' = M_n/\sigma$ where σ is the complex conjugation. Note that the action of σ on M_n is free if n is odd and has fixed points if n is even.

For $n \le 5$, the explicit topological type of M_n is known ([1], [2], [4], and [8]).

It is known that M_{2m+1} is a smooth manifold, while M_{2m} is a manifold with singular points ([5], [7], and [9]).

The purpose of this paper is to study the fundamental group of M_{2m} and M_{2m}'.

In [7], Y. Kamiyama and M. Tezuka showed by the Morse theory that if n is odd, the inclusion

$$i_n \colon M_n \to (S^1)^{n-1}$$

is a homotopy equivalence up to certain dimension.

Theorem 1.1 ([7]).

$$(i_{2m+1})_* \colon \pi_q(M_{2m+1}) \to \pi_q((S^1)^{2m})$$

is an isomorphism for $q \le m - 2$.

T.Hinokuma and H.Shiga showed the corresponding result for even n using the other Morse function ([3]). We give the alternative proof based on the method of [7] and show the following.

Theorem 1.2.

$$(i_{2m})_* \colon \pi_q(M_{2m}) \to \pi_q((S^1)^{2m-1})$$

is an isomorphism for $q \le m - 2$.

Remark 1.3. In [3], much more information about the topology of M_n is obtained.

In particular we have the following.

Corollary 1.4. $\pi_1(M_{2m})$ *is abelian for* $m \ge 3$.

The topology of M_n' is studied in [6] and he determined the fundamental group of M_{2m+1}' as well as almost all the homology groups of M_n'.

We determine the fundamental group of M_{2m}'.

Theorem 1.5. *For* $m \ge 3$

$$\pi_1(M_{2m}') = \mathbf{Z}/2.$$

2. Outline of the proof of Theorem 1.2. Following [7], we consider the function

$$g_{2m-1} \colon (S^1)^{2m-1} \to \mathbf{R}$$

defined by $g_{2m-1}(z_1,\ldots,z_{2m-1}) = |z_1+\ldots+z_{2m-1}-1|^2$. Note that $g_{2m-1}^{-1}(0) = M_{2m}$ is a "critical singular submanifold".

Proposition 2.1 ([7]).

$$(z_1,\ldots,z_{2m-1}) \in (S^1)^{2m-1} - M_{2m}$$

is a critical point of g_{2m-1} *if and only if* $z_i = \pm 1 \ (1 \le i \le 2m-1)$. *Moreover such points are non-degenerate with index greater than or equal to* m.

Proposition 2.2. *There exist* $0 < \varepsilon < 2$ *and a retraction* $r \colon g_{2m-1}^{-1}([0,\varepsilon]) \to M_{2m}$.

Combining these propositions, we see that $(i_{2m})_* \colon \pi_q(M_{2m}) \to \pi_q((S^1)^{2m-1})$ is injective for $q \le m - 2$. By [7], we know that $(i_{2m})_* \colon H_1(M_{2m};\mathbf{Z}) \to H_1((S^1)^{2m-1};\mathbf{Z})$ is an isomorphism, which complete the proof.

3. Outline of the proof of Theorem 1.5. We set

$$\Sigma_{n-1} = \{(z_1,\ldots,z_{n-1}) \mid z_i = \pm 1\} \subset (S^1)^{n-1}$$

$$\Sigma_{n-1}^1 = \{(z_1,\ldots,z_{n-1}) \mid z_i = \pm 1, \sum z_i = 1\} \subset \Sigma_{n-1}$$