Trigonal modular curves $X_0^{+d}(N)$

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1. Introduction. Let N be a positive integer, and let $X_0(N)$ be the modular curve corresponding to the congruence subgroup

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}) \;\middle|\; c \equiv 0 \bmod N \right\}.$$

In [8], we have determined the trigonal modular curves $X_0(N)$. Here an algebraic curve is said to be *trigonal* if it has a finite morphism of degree 3 to the projective line $\mathbf{P}^1$. According to [8], there are no non-trivial trigonal modular curves of type $X_0(N)$, that is, $X_0(N)$ is of genus at most 4 whenever it is trigonal. In this article, we determine the trigonal modular curves $X_0^{+d}(N) = X_0(N)/\langle W_d \rangle$ with $1 \neq d \| N$ (in case $d = N$ it is usually denoted by $X_0^+(N)$) by an argument analogous to [8]. The main result is

Theorem 1.

(i) *The curve $X_0^+(N)$ is trigonal of genus $g \geq 5$ if and only if*

$$N = 122, 146, 181, 227 \qquad (g = 5);$$
$$N = 164 \qquad (g = 6);$$
$$N = 162 \qquad (g = 7).$$

(ii) *If $d \neq N$, then $X_0^{+d}(N)$ is trigonal of genus $g \geq 5$ if and only if*

$$(N, d) = (147, 3) \qquad (g = 5);$$
$$(N, d) = (117, 13) \qquad (g = 6).$$

Consequently, it turns out that there do exist non-trivial trigonal modular curves of type $X_0^{+d}(N)$.

We shall prove this theorem only for $X_0^+(N)$. This is simply because we prefer to avoid the complexity of description. The argument of the next section will of course be applied without modification to the general case.

2. Determination of the trigonal modular curves $X_0^+(N)$. Let X be an algebraic curve of genus g. If $g \leq 2$, then it is trigonal; in fact, it is sub-hyperelliptic. Also, X is trigonal if it is non-hyperelliptic with $g = 3, 4$. On the other hand, any hyperelliptic curve of genus $g \geq 3$ is not trigonal. See [5] [1] or [8, § 1].

Let $W(N)$ be the group of Atkin–Lehner involutions on $X_0(N)$. All the pairs (N, W'), with W' a subgroup of $W(N)$, for which $X_0(N)/W'$ is hyperelliptic are determined by [6][7][4]. We record here a specific version.

Theorem 2. *The curve $X_0^+(N)$ has a hyperelliptic quotient curve of type $X_0(N)/W'$ of genus $g \geq 3$, if and only if*

$$N = 60, 66, 78, 85, 92, 94, 104, 105, 110, 120, 126,$$
$$136, 165, 171, 176, 195, 207, 252, 279, 315.$$

In particular, $X_0^+(N)$ itself is hyperelliptic of genus $g \geq 3$ if and only if

$$N = 60, 66, 85, 104 \qquad (g = 3);$$
$$N = 92, 94 \qquad (g = 4).$$

Given a non-negative integer g, it is not difficult to determine the values of N for which the genus $g^+(N)$ of $X_0^+(N)$ is equal to g. Thus we obtain:

Proposition 1. *The curve $X_0^+(N)$ is trigonal of genus $g = 3$ or 4 if and only if N is in the following list.*

g	N									
3	58	76	86	96	97	99	100	109	113	127
	128	139	149	151	169	179	239			
4	70	82	84	88	90	93	108	115	116	117
	129	135	137	147	155	159	161	173	199	215
	251	311								

From now on, we always assume $g^+(N) \geq 5$, and N is not in the list of Theorem 2. It is a fact that every trigonal curve over $\mathbf{Q}$ of genus $g \geq 5$ has a $\mathbf{Q}$-rational finite morphism of degree 3 to a rational curve over $\mathbf{Q}$ ([11, Thm. 2.1]). Therefore the argument of [8, § 3] is applicable. To be precise, fix a

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