A Thermodynamic formalism for one dimensional cellular automata

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(Communicated by Heisuke HIRONAKA, M.J.A., Feb. 12, 1999)

1. Introduction. The topological dynamics on the product space of a finite set $(S^{\mathbf{Z}}, \tau)$ is called one dimensional cellular automata (CA) if there exists a finite subset $\Lambda \subset \mathbf{Z}$ and a local map $f : S^{\Lambda} \mapsto S$ which satisfy $(\tau x)_i = f(x_{i+j}; j \in \Lambda)$ for all $x \in S^{\mathbf{Z}}$. Since its transient dynamics show various phenomena, the orbit structure is too much complicated to treat using typical method of the dynamical systems [2].

Let (X, σ) be the full shift over the symbol S. The dynamical zeta function with potential function $V \in BV(X)$ is defined as follows [1]:

$$\zeta(z, V) = \exp(\sum_{k>0} \frac{z^k}{k} Z_k(V)),$$
$$Z_k(V) = \sum_{x \in \mathrm{Fix}(X,\sigma^k)} e^{-S_k V(x)}$$

where $S_k V(x) = V(x) + V(\sigma x) + \cdots + V(\sigma^{k-1} x)$ and $\mathrm{Fix}(X, \sigma^k) = \{x \in X;\ \sigma^k x = x\}$.

Suppose that the potential function depends only one site x_0, i.e. $V(x) = V(x_0)$ for all $x = \{x_i\}_{i \in \mathbf{Z}}$, then the thermodynamical limit

$$P(V_n) = \lim_{k\to\infty} \frac{1}{k} \log Z_k(V_n),\ V_n(x) = V(\tau^n x)$$

exists for each n since $V_n(x)$ depends at most $np+1$ site.

The problem is the behavior of $P(V_n)$ as $n \to \infty$ and the relation with the transient dynamics of (X, τ).

In this paper we will show the structure of $\zeta(z, V_n)$ formally by the structure matrix of the local map $f^{(n)}$, where $f^{(n)} : S^{np+1} \mapsto S$ is defined naturally from f.

2. Results. At first, we define the $\#S^{np} \times \#S^{np}$ structure matrix $M_n(a)$, $a \in S$ with the index set is $\{r_1 \cdots r_{np};\ r_i \in S\}$.

Definition 2.1. The structure matrix $M_n(a)$ is as follows:

$$(M_n(a))_{r_1\cdots r_{np}, s_1\cdots s_{np}} = \begin{cases} 1 & \text{if } r_2 \cdots r_{np} = s_1 \cdots s_{np-1} \\ & \text{and } f^{(n)}(r_1 \cdots r_{np} s_{np}) = a, \\ 0 & \text{otherwise.} \end{cases}$$

We can express $Z_k(V_n)$ using trace formula of $M_n(a)$ as follows:

$$\begin{aligned}
Z_k(V_n) &= \sum_{x \in \mathrm{Fix}(X,\sigma^k)} e^{-S_k V(\tau^n x)} \\
&= \sum_{\substack{x\in \mathrm{Fix}(X,\sigma^k) \\ y \in \mathrm{Fix}(\tau^n X, \sigma^k) \\ \tau^n x = y}} e^{-S_k V(y)} d(y,k) \\
&\quad (d(x,k) = \#\{x \in \mathrm{Fix}(X,\sigma^k); \\
&\qquad\qquad \tau^n x = y, y \in \mathrm{Fix}(\tau^n X, \sigma^k)\}) \\
&= \sum_{y \in \mathrm{Fix}(\tau^n X,\sigma^k)} e^{-S_k V(y)} \\
&\qquad\qquad \mathrm{trace}(M_n(y_0) \cdots M_n(y_{k-1})) \\
&= \sum_{y \in \mathrm{Fix}(\tau^n X,\sigma^k)} \mathrm{trace}(e^{-V(y_0)} M_n(y_0) \\
&\qquad\qquad \cdots e^{-V(y_{k-1})} M_n(y_{k-1})) \\
&= \sum_{\substack{y_0 \in S \\ \cdots \\ y_{k-1} \in S}} \mathrm{trace}(e^{-V(y_0)} M_n(y_0) \\
&\qquad\qquad \cdots e^{-V(y_{k-1})} M_n(y_{k-1})) \\
&= \mathrm{trace}((\sum_{a \in S} e^{-V(a)} M_n(a))^k).
\end{aligned}$$

Thus, the zeta function is written by the determinant form.

$$\begin{aligned}
\zeta(z, V_n) &= \exp(\sum_{k>0} \frac{1}{k} \mathrm{trace}(z \sum_{a\in S} e^{-V(a)} M_n(a))^k) \\
&= \exp(\mathrm{trace}(\sum_{k>0} \frac{1}{k} (z \sum_{a \in S} e^{-V(a)} M_n(a))^k)) \\
&= \exp(\mathrm{trace}(-\log(I - z \sum_{a\in S} e^{-V(a)} M_n(a)))) \\
&= \det(I - z \sum_{a \in S} e^{-V(a)} M_n(a))^{-1} \\
&\quad (|z| < e^{-P(V_n)}).
\end{aligned}$$

As a result, we have the theorem.