A note on the mean value of the zeta and L-functions. IX

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The aim of the present note is to indicate the possibility of inverting our explicit formula [6, Theorem 4.2] for the fourth power moment of the Riemann zeta-function:

$$\mathcal{Z}_2(g) = \int_{-\infty}^{\infty} \left|\zeta\Big(\frac{1}{2}+it\Big)\right|^4 g(t)dt$$

where the weight g is to satisfy certain decay conditions.

To make this notion explicit we need first to introduce basic concepts from the theory of automorphic forms over the full modular group $\Gamma = PSL(2, \mathbf{Z})$ (see [6, Chapters 1–3]). Thus, let $\{1/4+\kappa_j^2\}$ $(\kappa_j > 0; j = 1, 2, \ldots)$ be the discrete spectrum of the hyperbolic Laplacian $\Delta = -y^2((\partial/\partial x)^2 + (\partial/\partial y)^2)$ acting over the Hilbert space composed of all Γ-automorphic functions which are square integrable with respect to the hyperbolic measure. Let $\{\psi_j\}$ be a maximal orthonormal system such that $\Delta\psi_j = (1/4+\kappa_j^2)\psi_j$ for each $j \geq 1$ and $T(n)\psi_j = t_j(n)\psi_j$ for each integer $n \geq 1$, where

$$(T(n)f)(z) = n^{-1/2}\sum_{ad=n}\sum_{b=1}^{d} f\Big(\frac{az+b}{d}\Big)$$

is the Hecke operator. We may further assume that $\psi_j(-\bar{z}) = \epsilon_j\psi_j$ with $\epsilon_j = \pm 1$. We then define

$$H_j(s) = \sum_{n=1}^{\infty} t_j(n)n^{-s},$$

which we call the Hecke series associated with ψ_j, and which can be continued to an entire function.

The integral $\mathcal{Z}_2(g)$ is expanded over the spectra of Δ, yielding the explicit formula under present consideration. Its part $\mathcal{Z}_{2,d}(g)$ corresponding to the discrete spectrum has the following expression:

$$\mathcal{Z}_{2,d}(g) = \sum_{j=1}^{\infty} \alpha_j H_j\left(\frac{1}{2}\right)^3 \Lambda(\kappa_j; g), \tag{1}$$

where $\alpha_j = |\rho_j|^2/\cosh(\pi\kappa_j)$ with the first Fourier coefficient ρ_j of ψ_j, and

$$\begin{aligned}\Lambda(r;g) = \mathrm{Re}\Big[\Big(1+\frac{i}{\sinh(\pi r)}\Big) &\times \int_0^{\infty} (y(1+y))^{-1/2} g_c\Big(\log\Big(1+\frac{1}{y}\Big)\Big)\frac{\Gamma(1/2+ir)^2}{\Gamma(1+2ir)} \\ &\times F\Big(\frac{1}{2}+ir, \frac{1}{2}+ir; 1+2ir; -\frac{1}{y}\Big)y^{-1/2-ir}dy\Big]\end{aligned} \tag{2}$$

with

$$g_c(x) = \int_{-\infty}^{\infty} g(t)\cos(xt)dt,$$

and F the hypergeometric function.

We regard $\Lambda(r;g)$ as an integral transform of g. Then we claim that for a given h satisfying appropriate conditions one may find a g such that

$$\Lambda(r;g) = \mathrm{Re}\Big[\Big(1+\frac{i}{\sinh(\pi r)}\Big)h(r)\Big]. \tag{3}$$

Since other parts of the explicit formula for $\mathcal{Z}_2(g)$ have structures essentially the same as (1), this solution g gives rise to the notion stated at the beginning.

We shall thus solve the integral equation (3). Our discussion will, however, be formal in the sense that convergence issues and necessary estimations are all skipped. This is to show the core of our idea, which is in fact very simple: We begin with the observation that the expression (2) can be written as

$$\begin{aligned}\Lambda(r;g) = 2\mathrm{Re}\Big[\Big(1+\frac{i}{\sinh(\pi r)}\Big)&\int_0^{\infty}(y(1+y))^{-1/2} \\ &\times g_c\Big(\log\Big(1+\frac{1}{y}\Big)\Big)Q_{-1/2+ir}(1+2y)dy\Big],\end{aligned}$$

where $Q_\nu(z)$ is the Legendre function of order ν of the second kind. We have used the relation

$$\begin{aligned}Q_\nu(z) = &\frac{\sqrt{\pi}\Gamma(\nu+1)}{2^{\nu+1}\Gamma(\nu+3/2)} \\ &\times(z-1)^{-\nu-1}F\Big(\nu+1, \nu+1; 2\nu+2; \frac{2}{1-z}\Big),\end{aligned}$$

which holds, say, for $z > 1$ and $\nu \neq -1, -2, \ldots$ (via analytic continuation applied to the first formula in the problem 4 on [3, p. 200]). Namely, the equation