"Hasse principle" for $GL_2(D)$

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1. Statement of a theorem. Let D be a Euclidean domain and $G = GL_2(D)$, the group of invertible 2×2 matrices over D.[0] We shall prove that

(1.1) **Theorem.** $Ш(G) = 1$, *i.e., G enjoys the "Hasse principle"*.[1]

(1.2) **Remark.** Thanks to an excellent idea of M. Mazur, to prove (1.1) it is enough to verify that

$$\mathrm{End}_c(G) = \mathrm{Inn}(G), \tag{1.3}$$

where the left hand side is the set of all endomorphisms of G preserving conjugacy classes of G.[1] Thus for each $F \in \mathrm{End}_c(G)$, and $A \in G$, we have

$$F(A) \sim A, \quad \text{i.e.,} \quad F(A) = PAP^{-1}, \quad P \text{ depending on } A. \tag{1.4}$$

Given an $F \in \mathrm{End}_c(G)$ we connect two elements A, B of G by *a* string according to the rule:

$$A \,\text{—}\, B \Longleftrightarrow \exists P \in G \quad \text{so that} \quad F(A) = PAP^{-1} \text{ and } F(B) = PBP^{-1}. \tag{1.5}$$

Note that A — B is not, a priori, an equivalence relation defined on G.[2] Even so, this relation is very useful to prove the Hasse principle $Ш(G) = 1$. Note also that the relation (1.5) depends only on F modulo $\mathrm{Inn}(G)$.

2. Generators for G. Before proving (1.1), let us gather some basic facts on $G = GL_2(D)$, D being a Euclidean domain. Denote by D^* the group of invertible elements of D. Let N, M_λ ($\lambda \in D$, $\lambda \neq 0$), D_μ ($\mu \in D^*$, $\mu \neq 1$) be elements of G defined by

$$N = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad M_\lambda = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \quad D_\mu = \begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix}. \tag{2.1}$$

It is well-known and easy to prove that

$$G \text{ is generated by } N,\ M_\lambda,\ D_\mu: \quad G = \langle N, M_\lambda, D_\mu \rangle, \tag{2.2}$$

We will use repeatedly the following equalities on $P = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in G$.

$$PNP^{-1} = (\det P)^{-1} \begin{pmatrix} yt - xz & x^2 - y^2 \\ t^2 - z^2 & xz - yt \end{pmatrix}, \tag{2.3}$$

$$M_\lambda PNP^{-1} = (\det P)^{-1} \times \begin{pmatrix} yt - xz + \lambda(t^2 - z^2) & x^2 - y^2 + \lambda(xz - yt) \\ t^2 - z^2 & xz - yt \end{pmatrix}, \tag{2.4}$$

$$D_\mu PNP^{-1} = (\det P)^{-1} \begin{pmatrix} yt - xz & x^2 - y^2 \\ \mu(t^2 - z^2) & \mu(xz - yt) \end{pmatrix}. \tag{2.5}$$

3. Proof of the theorem.

Step (I). To prove that N — M_λ. Since we can adjust a given F in $\mathrm{End}_c(G)$ by elements of $\mathrm{Inn}(G)$, We may assume that

$$\begin{cases} F(M_\lambda) = M_\lambda, \\ F(N) = PNP^{-1}, \quad P \in G. \end{cases} \tag{3.1}$$

Our problem is to find $P_0 \in G$ so that

$$\begin{cases} F(M_\lambda) = M_\lambda = P_0 M_\lambda {P_0}^{-1}, \\ F(N) = PNP^{-1} = P_0 N {P_0}^{-1}. \end{cases} \tag{3.2}$$

Put

$$P = \begin{pmatrix} x & y \\ z & t \end{pmatrix}, \quad P_0 = \begin{pmatrix} 1 & y_0 \\ 0 & 1 \end{pmatrix}. \tag{3.3}$$

Clearly P_0, with any $y_0 \in D$, meets the first equality of (3.2). As for the second equality of (3.2), in view of (2.3) for P and P_0 we are forced to set $y_0 = (yt - xz)/(xt - yz)$ and then we should verify the equality (3.2) which boils down to a single equality:

$$\det(P) = xt - yz = t^2 - z^2 \tag{3.4}$$

as a little calculation shows. To get (3.4), we must use seriously the assumption that F is a homomor-

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[0] Needless to say, D may be any commutative field.

[1] As for unexplained notation and facts in this paper, see [1].

[2] This reminds me somehow a children's string game CAT'S CRADLE, or AYATORI in Japanese. One can play this game on any group G once an endomorphism $F \in \mathrm{End}_c(G)$ is chosen.