A note on algebraic aspects of boundary feedback control systems of parabolic type

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1. Introduction. In the study of stabilization of boundary control systems, most fundamental is the static feedback control scheme: Based on a finite number of the observed data (outputs), it is the scheme to feed them back *directly* into the system through the boundary. Let Ω denote a bounded domain of $\mathbb{R}^m$ with the boundary Γ which consists of a finite number of smooth components of $(m-1)$-dimension. The control system studied here is the following initial-boundary value problem:

$$
\begin{aligned}
&\frac{\partial u}{\partial t} + \mathcal{L}u = 0 \quad \text{in } (0,\infty)\times\Omega,\\
(1)\qquad &\tau u = \sum_{k=1}^{N} \langle u, w_k\rangle_{\Omega}\, h_k \quad \text{on } (0,\infty)\times\Gamma,\\
&u(0,\cdot) = u_0(\cdot) \quad \text{in } \Omega.
\end{aligned}
$$

Here, $\mathcal{L}$ denotes a uniformly elliptic differential operator of order 2 in Ω defined by

$$
\mathcal{L}u = -\sum_{i,j=1}^{m} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j}\Big) + \sum_{i=1}^{m} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u,
$$

and $a_{ij}(x) = a_{ji}(x)$ for $1 \leqslant i,\ j \leqslant m,\ \ x \in \overline{\Omega}$. The boundary operator τ associated with $\mathcal{L}$ is either τ_1 of the Dirichlet type or τ_2 of the Robin type:

$$
\begin{aligned}
\tau_1 u &= u|_{\Gamma},\\
\tau_2 u &= \frac{\partial u}{\partial \nu} + \sigma(\xi)u\\
&= \sum_{i,j=1}^{m} a_{ij}(\xi)\nu_i(\xi)\frac{\partial u}{\partial x_j}\Big|_{\Gamma} + \sigma(\xi)u|_{\Gamma},
\end{aligned}
$$

where $(\nu_1(\xi), \ldots, \nu_m(\xi))$ denotes the unit outer normal at $\xi \in \Gamma$. Necessary regularity on $\overline{\Omega}$ and on Γ of coefficients of $\mathcal{L}$ and τ is assumed tacitly. The inner product and the norm in $L^2(\Omega)$ are denoted by $\langle\cdot,\cdot\rangle_{\Omega}$ and $\|\cdot\|$, respectively. The symbol $\|\cdot\|$ is also used for the $\mathcal{L}(L^2(\Omega))$-norm. In eq. (1), $\langle u, w_k\rangle_{\Omega}$ denote the outputs, where $w_k \in L^2(\Omega)$, and h_k the actuators belonging to $H^{3/2}(\Gamma)$ in the case of the Dirichlet boundary condition, or $H^{1/2}(\Gamma)$ in the Robin boundary condition.

Let us define the linear operators L_i and M_i, $i = 1, 2$ in $L^2(\Omega)$ by

$$
\begin{aligned}
&L_i u = \mathcal{L}u, \quad u \in \mathcal{D}(L_i),\\
&\mathcal{D}(L_i) = \{u \in H^2(\Omega)\,;\ \tau_i u = 0 \text{ on } \Gamma\}
\end{aligned}
$$

and

$$
\begin{aligned}
&M_i u = \mathcal{L}u, \quad u \in \mathcal{D}(M_i),\\
&\mathcal{D}(M_i) = \Big\{u \in H^2(\Omega)\,;\\
&\qquad\qquad \tau_i u = \sum_{k=1}^{N}\langle u, w_k\rangle_{\Omega}\, h_k \text{ on } \Gamma\Big\},
\end{aligned}
$$

respectively. Henceforth L stands for either L_1 or L_2 when it is distinguished from the context. The same symbolic convention applies to M_i as well as other operators. Eq. (1) is then simply rewritten as the equation in $L^2(\Omega)$:

$$
(2)\qquad \frac{du}{dt} + Mu = 0, \quad u(0) = u_0.
$$

Given a $\mu > 0$, the problem is to find w_k's and h_k's such that the semigroup $\exp(-tM)$ satisfies the decay estimate

$$
(3)\qquad \|e^{-tM}\| \leqslant \text{const}\, e^{-\mu t}, \quad t \geqslant 0.
$$

In [4], this estimate was established via the fractional powers L_c^{ω}, $L_c = L + c$, $c > 0$ and the related fractional calculus. In the case of the Robin boundary condition, for example, we set

$$
x(t) = L_{2c}^{-\omega}u(t), \quad \frac{1}{4} < \omega < \frac{1}{2},
$$

and, noticing the relation: $\mathcal{D}(L_{2c}^{\omega}) = H^{2\omega}(\Omega)$ for