The CM-fields with Class Number One which are Hilbert Class Fields of Quadratic Fields

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It is known that there are only finitely many normal CM-fields with class number one (see [3]). Moreover, all the imaginary abelian number fields with call number one have been determined by K. Yamamura: There are 172 such fields. Hence, it is time to determine all the non-abelian normal CM-fields with class number one. The author and R. Okazaki in [7] found all the non-abelian normal octic CM-fields with class number one: There are 17 such fields, all of which are dihedral and narrow Hilbert class fields of real quadratic fields. Then, the author, R. Okazaki and M. Olivier in [4] found all the non-abelian normal CM-fields of degree 12 with class number one: There are 9 such fields, all of which are dehedral. We shall explain in this note how we determine in [6] all CM-fields $\boldsymbol{H}_s(\boldsymbol{k})$ with class number one (without fixing their degrees) which are narrow Hilbert class fields of quadratic fields $\boldsymbol{k}$: There are at least 95 such fields, and assuming a conjecture put forward in [5], there are exactly 95 such fields. All the proofs of the results stated here may be found in [6]. Note that 4 of them have degree 16,1 of them has degree 20 and 1 of them has degree 24, these six fields being non-abelian normal dihedral CM-fields.

We let $\boldsymbol{G}_{\boldsymbol{k}}$ be the genus field of $\boldsymbol{k}$, which is the maximal abelian number field that is unramified over $\boldsymbol{k}$ at all the finite places. In this paper, genus field will always stand for genus field of some quadratic field. The Hilbert class field (in larger sense) $\boldsymbol{H}_l(\boldsymbol{k})$ of a real quadratic field is totally real. Hence, it is not a CM-field. However, one can easily prove that the narrow Hilbert class field $\boldsymbol{H}_s(\boldsymbol{k})$ of a real quadratic field $\boldsymbol{k}$ is a CM-field if and only if the norm of the fundamental unit of $\boldsymbol{k}$ is equal to $+1$, which we will assume for the rest of this paper. In that case, $\boldsymbol{H}_l(\boldsymbol{k})$ is the maximal totally real subfield of $\boldsymbol{H}_s(\boldsymbol{k})$.

1. Let us first focus on Hilbert class fields of imaginary quadratic fields.

Theorem 1 (See [2, proof of Th. 6.1]). *Let $\boldsymbol{k}$ be an imaginary quadratic field. Let $\boldsymbol{H}_s(\boldsymbol{k})$ be the Hilbert class field of $\boldsymbol{k}$. Then, $\boldsymbol{H}_s(\boldsymbol{k})$ is Galois and its Galois group* $\mathrm{Gal}(\boldsymbol{H}_s(\boldsymbol{k})/\boldsymbol{Q})$ *is a generalized dihedral group which is a semi-direct product of* $\boldsymbol{A} = \mathrm{Gal}(\boldsymbol{H}_s(\boldsymbol{k})/\boldsymbol{k})$ *which is canonically isomorphic to the ideal class group of $\boldsymbol{k}$ with* $\{\mathrm{Id}, c\}$ *where the complex conjugation c acts on $\boldsymbol{A}$ via $cac^{-1} = a^{-1}$.*

Now, an imaginary normal number field is a CM-field if and only if the complex conjugation is in the center of its Galois group. Hence, the Hilbert class field $\boldsymbol{H}_s(\boldsymbol{k})$ of an imaginary quadratic field $\boldsymbol{k}$ is a CM-field if and only if the ideal class group of $\boldsymbol{k}$ has exponent ≤ 2, in which case $\boldsymbol{H}_s(\boldsymbol{k}) = \boldsymbol{G}_{\boldsymbol{k}}$. Hence, we have only to determine all imaginary genus fields $\boldsymbol{G}_{\boldsymbol{k}} = \boldsymbol{Q}(\sqrt{p_1^*}, \cdots, \sqrt{p_t^*})$ with class number one. Here, $\boldsymbol{k} = \boldsymbol{Q}(\sqrt{p_1^* p_2^* \cdots p_t^*})$ has discriminant $p_1^* p_2^* \cdots p_t^*$, where $p_1, p_2, \cdots, p_t$ are $t \geq 1$ distinct primes and

$$p_i^* = \begin{cases} p_i & \text{if } p_i \equiv 1 \pmod 4, \\ -p_i & \text{if } p_i \equiv 3 \pmod 4, \\ -4,\ -8 \text{ or } 8 & \text{if } p_i \equiv 2. \end{cases}$$

Note that $\boldsymbol{G}_{\boldsymbol{k}}$ has degree 2^t. In fact, it is easy to determine all imaginary genus fields with relative class number one, and then to compute the class numbers of their maximal real subfields. To this end, we prove that if $\boldsymbol{G}$ and $\boldsymbol{G}'$ are two imaginary genus fields such that $\boldsymbol{G} \subseteq \boldsymbol{G}'$ then the relative class number of $\boldsymbol{G}$ divides that of $\boldsymbol{G}'$, and if $\boldsymbol{G}_{\boldsymbol{k}}$ has odd relative class number, then $t \leq 3$. As all the imaginary quadratic fields with class number ≤ 2 are known, we get (see [6]):

Theorem 2. *There are exactly* 73 *imaginary genus fields $\boldsymbol{G}_{\boldsymbol{k}} = \boldsymbol{Q}(\sqrt{p_1^*}, \cdots, \sqrt{p_t^*})$ with relative class number one: those which appear in the follow-*

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