Recurrence and Transience of Operator Semi-Stable Processes

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1. Introduction and results. Operator semi-stable distributions on the d-dimensional Euclidean space $\boldsymbol{R}^d$ constitute a class of infinitely divisible distributions. They are studied by R. Jajte [4], [5], W. Krakowiak [6], A. Luczak [9], V. Chorny [2] and others. We call Lévy processes on $\boldsymbol{R}^d$ having operator semi-stable distributions at each time *operator semi-stable processes*. Here we mean by Lévy processes stochastically continuous processes with stationary independent increments starting at the origin. In this note we determine recurrence and transience of all non-degenerate operator semi-stable processes.

A distribution μ on $\boldsymbol{R}^d$ is called *operator semi-stable* if there exist a sequence $\{Y_n : n = 1,2,\cdots\}$ of i.i.d. (= independent identically distributed) random variables on $\boldsymbol{R}^d$, a strictly increasing sequence of positive integers k_n satisfying $k_{n+1}/k_n \to r$ with some $r \in [1, \infty)$, and sequences of invertible linear operators A_n acting in $\boldsymbol{R}^d$ and vectors b_n in $\boldsymbol{R}^d$ such that the distribution of

(1.1) $$A_n(Y_1 + Y_2 + \cdots + Y_{k_n}) + b_n$$

weakly converges to μ as $n \to \infty$. R. Jajte [4] shows that if μ is full (that is, the support of μ is not contained in any $(d-1)$-dimensional hyperplane in $\boldsymbol{R}^d$), then a necessary and sufficient condition for μ to be operator semi-stable is that it is infinitely divisible and there exist a number $a \in (0, 1)$, a vector $b \in \boldsymbol{R}^d$, and an invertible linear operator A such that

(1.2) $$\mu^a = A\mu * \delta_b.$$

Here μ^a is the a-th convolution power of μ, $A\mu$ is the distribution defined by $A\mu(E) = \mu(A^{-1}E)$, and δ_b is the delta distribution at b. Using the relation (1.2), A. Luczak [9] and V. Chorny [2] describe the Lévy measure of μ.

In one dimension $(d = 1)$ A_n and A are simply multiplication by non-zero constants. P. Lévy [8], p. 204, introduced in one dimension the notion of semi-stability, which corresponded to the case $b = 0$ in (1.2), and determined their characteristic functions. R. Shimizu[13] made a study of relations of Lévy's semi-stability with limit theorems for sequences of i.i.d. random variables. V. M. Kruglov [7] studied the class of one-dimensional distributions which are limit distributions of $c_n(Y_1 + Y_2 + \cdots + Y_{k_n}) + b_n$ for i.i.d. random variables $\{Y_n\}$ with $c_n > 0$, b_n real, and $k_{n+1}/k_n \to r \in [1, \infty)$. In general finite dimensions, if $k_n = n$ and A_n is a positive constant multiple of the identity operator for each n, then the definition above of operator semi-stability gives the class of stable distributions on $\boldsymbol{R}^d$. If $k_n = n$, then the definition above gives the class of operator stable distributions, which were first introduced by M. Sharpe [12]. On the other hand, if A_n is a non-zero constant multiple of the identity operator for each n, then the limit distributions are called semi-stable. The class of operator semi-stable distributions extends these classes. The corresponding Lévy processes are called stable processes, operator stable processes, semi-stable processes, and operator semi-stable processes, respectively. Classification of stable processes into recurrent and transient is well-known. It is extended in [1] to semi-stable processes. Operator stable processes are discussed in [10], but their recurrence and transience are not treated.

Our result is as follows. We say that a Lévy process is *non-degenerate* if its distribution at each $t > 0$ is full.

Theorem. *Let $\{X_t\}$ be a non-degenerate operator semi-stable process on the plane $\boldsymbol{R}^2$. If $\{X_t\}$ is not Gaussian, then it is transient.*

Note that, for $d > 3$, all non-degenerate Lévy processes on $\boldsymbol{R}^d$ are transient (see [11] for proof). Also note that operator semi-stable processes on the line $\boldsymbol{R}^1$ are semi-stable processes in the sense of [1], and their classification is obtained in [1]. A Gaussian Lévy process on the plane $\boldsymbol{R}^2$ is recurrent or transient according as its mean is zero or non-zero, respectively. There-