Geometric Transition for a Class of Hyperbolic Operators with Double Characteristics

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1. Introduction. The Cauchy problem in the C^∞ category for hyperbolic operators with double characteristics has been studied by many authors under different geometrical perspectives. These results (see e.g. [15], [1] and the references therein) put in evidence the role of both the lower order terms and the symplectic geometry associated with the principal symbol at the double characteristic points.

The purpose of our work is the study of a case in which the set of double points $\sum_2$ can be written as the reunion of two nonempty subsets $\sum_{2,e}$, $\sum_{2,ne}$ i.e. the set of effectively hyperbolic double points and its $\sum_2$-complementary respectively. We say that in this case a "transition" occurs, or, roughly speaking, that a real eigenvalue of the fundamental matrix defined on $\sum_{2,e}$ vanishes on $\sum_{2,ne}$.

Trying to provide a unifying approach to the first of the open problems mentioned above, R. B. Melrose, [15], in 1984 formulated the following conjecture:

Conjecture. *If the Cauchy problem for P (having at most double characteristics) is well-posed in Ω_0 then $s(\rho)/\lambda(\rho)$ is uniformly bounded in $\sum_{2,e} \cap T^*V\setminus\{0\}$, for some neighborhood V of $0 \in \boldsymbol{R}^{n+1}$, where*

$$s(\rho) = |\operatorname{Im} P^s_{m-1}(\rho)| + \inf\{|\operatorname{Re} P^s_{m-1}(\rho) - s|\,;\, |s| \leq \operatorname{Tr}^+ F_{P_m}(\rho)\}.$$

This conjecture, as it stands, has not been proved yet, however it gives a hint of what one should reasonably look for, when trying to formulate Levi conditions on the subprincipal symbol P^s_{m-1}.

The purpose of this paper is actually to understand a degenerate effectively hyperbolic case: more precisely we will show that the uniform boundedness of the ratio $s(\rho)/\lambda(\rho)$ may also serve as a sufficient condition for the well-posedness of the Cauchy problem, provided certain symplectically invariant geometric conditions are satisfied. Our Assumption (H4) is therefore a slightly restricted reformulation of Melrose Conjecture, and this is expected since we are dealing with a sufficiency result.

Even under the usual assumption that the principal symbol P_m vanishes exactly of order two on $\sum_2$, one readily sees that the nature of the set $\sum_{2,ne}$ (henceforth denoted by $\sum_2'$), the order of vanishing of λ on $\sum_{2,ne}$ and the geometry of the transition from a symplectic point of view can be of a very wild type and may produce nontrivial situations.

Therefore we started assuming that $\sum_2$ is a smooth submanifold of $T^*\Omega\setminus\{0\}$. Furthermore in Section 3 below we collected a number of results serving as a symplectic classification of the transition cases that can possibly occur. In this framework Assumption (H2) isolates one of these cases.

More precisely (H2) takes a picture of $F_{P_m}(\rho)$ when ρ is inside or outside $\sum_2'$, avoiding for instance Jordan blocks of size 4 in the canonical form of $F_{P_m}(\rho)$, $\rho \in \sum_2'$ and precising that $0 \leq \lambda(\rho) \leq \operatorname{dist}^2_{\sum_2}(\rho, \sum_2')$, where dist denotes any geodesic distance of $\rho \in \sum_2$ from $\sum_2'$; but this is not yet enough to prove an energy estimate eventually leading to existence and uniqueness. Indeed we recall that in the non degenerate effectively hyperbolic case (see e.g. Lemma 1.2.1 in [17]), denoting by $\Gamma_{P_m}(\rho)$ the hyperbolicity cone of the localized hyperbolic quadratic form $\sigma(X, F_{P_m}(\rho)X)$, "$P_m$ is effectively hyperbolic at ρ" if and only if "$\Gamma_{P_m}(\rho) \cap \operatorname{ran} F_{P_m}(\rho) \cap [(0, e_0)]^\sigma \neq \emptyset$, where with $V^\sigma = \{z \in T_\rho T^*\Omega \mid \sigma(z, v) = 0,\ \forall\, v \in V\}$ we define the dual with respect to the symplectic form σ of a vector subspace V of $T_\rho T^*\Omega$.

Therefore for an effectively hyperbolic operator it is always possible to find a time function f, i.e. a C^∞ function vanishing on the double set of P_m, whose Hamilton vector field $H_f(\rho)$, $\rho \in \sum_2$ belongs to $\Gamma_{P_m}(\rho)$ (and for which in addition