# 57. *On Fractional Powers of a Class of Elliptic Differential Operators with Feedback Boundary Conditions. II*

By Takao NAMBU

Department of Applied Mathematics, Faculty of Engineering, Kobe University

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**§1. Introduction.** In the study of boundary control systems, fractional powers of elliptic differential operators are of special importance. They often appear in optimal control and stabilization problems, and play a central role there, We consider in this paper a system of differential operators $(\mathscr{L}, \tau)$ in a bounded domain $\Omega$ of $\mathbb{R}^m$ with the boundary $\Gamma$ which consists of a finite number of smooth components of $(m-1)$-dimension. Actually, let $\mathscr{L}$ denote a uniformly elliptic differential operator of order 2 in $\Omega$ defined by

$$\mathscr{L}u = -\sum_{i,j=1}^{m} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{m} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u,$$

where $a_{ij}(x) = a_{ji}(x)$ for $1 \le i, j \le m$ and $x \in \bar{\Omega}$. Associated with $\mathscr{L}$ is a boundary operator $\tau$ of the Neumann or Robin type given by

$$\tau u = \frac{\partial u}{\partial \nu} + \sigma(\xi)u = \sum_{i,j=1}^{m} a_{ij}(\xi)\nu_i(\xi)\frac{\partial u}{\partial x_j} + \sigma(\xi)u,$$

where $(\nu_1(\xi), \dots, \nu_m(\xi))$ denotes the unit outer normal at $\xi \in \Gamma$. Necessary regularity on $\bar{\Omega}$ and on $\Gamma$ of coefficients of $\mathscr{L}$ and $\tau$ is assumed tacitly. Moreover $\sigma(\xi)$ is assumed to have a suitable smooth extension to $\bar{\Omega}$. Let us define the linear operators $L$ and $M$ in $L^2(\Omega)$ by

$$Lu = \mathscr{L}u, \quad u \in \mathscr{D}(L) = \{u \in H^2(\Omega)\,;\, \tau u = 0 \text{ on } \Gamma\}$$

and

$$Mu = \mathscr{L}u, \quad u \in \mathscr{D}(M) = \left\{u \in H^2(\Omega)\,;\, \tau u = \sum_{k=1}^{p} \langle u, w_k\rangle_\Gamma h_k \text{ on } \Gamma\right\},$$

respectively. Here, $w_k \in L^2(\Gamma)$ stand for weight functions of observations distributed on $\Gamma$; $h_k$ the actuators belonging to $H^{1/2}(\Gamma)$; $\langle\cdot\,,\cdot\rangle_\Gamma$ the inner product in $L^2(\Gamma)$; and $p$ a positive integer depending on the control problems under consideration. Thus the boundary condition for $M$ may be described as a feedback type. The operator $M$ is not a standard type in the sense that the boundary condition is composed of terms of local nature and those of global nature. All norms hereafter will be $L^2(\Omega)$-or $\mathscr{L}(L^2(\Omega))$-norms unless otherwise indicated. As is well known [7], there is a sector $\overline{\Sigma}_{-\alpha} = \overline{\Sigma} - \alpha$, $\alpha > 0$, such that $\overline{\Sigma}_{-\alpha}$ is contained in the resolvent set $\rho(L)$, where $\overline{\Sigma} = \{\lambda\,;\, \theta \le |\arg\lambda| \le \pi\}$, $0 < \theta < \pi/2$, and the upper bar means the closure of a set. Choose a positive constant $c(> \alpha)$, and let $L_c = L + c$. Then fractional powers of the operator $L_c$ are well defined. As is well known [2], we have the characterization of $L_c^\omega$