88. *Euler Characteristics of Groups and Orbit Spaces of Free G-complexes*

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Abstract: Let Γ be a group of finite homological type, X a finite dimensional, free Γ-complex such that $H_*(X, \mathbf{Z})$ is finitely generated. We proved that $H_*(X/\Gamma, \mathbf{Z})$ is finitely generated, and $\chi(X/\Gamma) = \chi(\Gamma)\cdot \chi(X)$, where $\chi(\Gamma)$ is the Euler characteristic of a group Γ.

Key words: Topological transformation groups; Euler characteristics of groups.

Let Γ be a discrete group. By a *Γ-complex* we will mean a CW-complex X together with an action of Γ on X which permutes the cells. A Γ-complex X is said to be *free* if the action of Γ freely permutes the cell of X. This note attempts to provide a formula concerning the Euler characteristics of a finite dimensional, free Γ-complex X and of the orbit space X/Γ. Such a formula is well-known for finite groups [3, p. 245]:

Theorem 1. *Let G be a finite group, X a finite dimensional free G-complex such that $H_*(X, \mathbf{Z})$ is finitely generated. Then $H_*(X/G, \mathbf{Z})$ is finitely generated, and*

$$\chi(X) = |G| \cdot \chi(X/G).$$

On the other hand, let Γ be a fundamental group of a finite aspherical complex $B\Gamma$. Then we see that $H_*(X/\Gamma, \mathbf{Z})$ is finitely generated, and $\chi(X/\Gamma) = \chi(B\Gamma)\cdot\chi(X)$. This follows from the product formula of Euler characteristics for fibrations applied to the Borel construction $X \to E\Gamma\times_\Gamma X \to B\Gamma$, where $E\Gamma$ is the universal cover of $B\Gamma$, and from the fact that $H_*(E\Gamma\times_\Gamma X, \mathbf{Z}) \cong H_*(X/\Gamma, \mathbf{Z})$. We will unify and extend these formulae by means of the Euler characteristics of groups.

The Euler characteristics of abstract groups has been studied by a number of authors under different conditions. We employ the one developed in Brown's book [3]. Recall that Γ is a *group of finite homological type* if (i) Γ is a group of finite virtual cohomological dimension (written $\operatorname{vcd}\Gamma < \infty$) and (ii) for any $\mathbf{Z}\Gamma$-module M which is finitely generated as a $\mathbf{Z}$-module, $H_n(\Gamma, M)$ is finitely generated for all n. A group Γ is of finite homological type if and only if a subgroup of finite index is.

Given a group Γ of finite homological type, the *Euler characteristic* $\chi(\Gamma)$ of Γ is defined. Namely, when Γ is torsion-free, set

$$\chi(\Gamma) = \sum_i (-1)^i \operatorname{rank}_{\mathbf{Z}} H_i(\Gamma, \mathbf{Z}).$$

When Γ has torsion, set