84. *Orders in Quadratic Fields. II*

By R. A. MOLLIN*) and L.-C. ZHANG**)

(Communicated by Shokichi IYANAGA, M. J. A., Nov. 12, 1993)

Abstarct: We provide very sharp lower bounds for the class numbers of arbitrary complex quadratic order.

Key words: Complex quadratic order; class number; quadratic polynomial.

The work herein continues that of [5] to which we refer the reader for background information and notation. This also complements the work of the authors in [6] where we dealt with the real case.

Our principal result (Theorem 1 below) provides a sharp lower bound for h_Δ when $\Delta < 0$ is the discriminant of any complex quadratic order, and yields as a consequence a complete generalization of the well-known result by Rabinowitsch [8] for $h_{\Delta_0} = 1$, and includes the more recent result by Sasaki [9] for $h_{\Delta_0} = 2$. Furthermore, our results yield sharper bounds than those given heretofore in the literature such as Oesterlé [7] and Buhler, Gross and Zagier [2]. Most recently Sasaki [9] gave the following lower bound

$$(*) \qquad h_{\Delta_0} \geq d(N(b+\omega))$$

where b is any non-negative integer with $b \leq |\Delta_0|/4 - 1$ and $d(m)$ is the number of (not necessarily distinct) prime divisors of m.

It is in the context of $(*)$ that we couch our main result which will be seen to be a much sharper bound as follows. In the following $D = f^2 D_0$ where D_0 is the radicand of $\boldsymbol{Q}(\sqrt{\Delta}) = \boldsymbol{Q}(\sqrt{D_0})$.

Theorem 1. *Let $\Delta < 0$ be a discriminant with odd conductor f. If b is any integer and M is any divisor of $N(b+\omega_\Delta)$ with $M < N(\omega_\Delta)$ and $gcd(M, f) = 1$ then $h_\Delta \geq \tau(M)$, the number of distinct positive divisors of M.*

Proof. It suffices to show that if $a_1 \neq a_2$ are both divisors of M then $I_1 = [a_1, b+\omega_\Delta]$ is not equivalent to $I_2 = [a_2, b+\omega_\Delta]$. Suppose, to the contrary that $I_1 \sim I_2$.

Claim. There exist relatively prime integers x and y satisfying

$$(1) \qquad ((\sigma a_1 x) + (\sigma b + \sigma - 1)y)^2 - Dy^2 = \sigma^2 a_1 a_2.$$

$$(2) \qquad a_2 \mid (a_1 x + (2b + \sigma - 1)y).$$

$$(3) \qquad \sigma^2 a_1 a_2 \mid (D - (\sigma b + \sigma - 1)^2) y.$$

We only prove the case where $\sigma = 1$ since the other case is similar. Since $I_1 \sim I_2$ then there exists an element $\gamma \in I_1$ such that $(\gamma) I_2 = (a_2) I_1$

---

*) Department of Mathematics and Statistics, The University of Calgary, Canada.
The first author's research is supported by NSERC Canada grant # A8484.
**) Mathematics Department, Southwest Missouri State University, U. S. A.
The second author's research is supported by an SMSU Faculty Summer Fellowship.