65. *An Application of a Theorem of Rudin*

By J. C. SMITH

Virginia Technological Institute and State University

(Communicated by Shokichi IYANAGA, M. J. A., Nov. 12, 1992)

1. Introduction. The study of generalized paracompact spaces has become significant in recent years. In addition to the new results in this area there have been a number of new interesting questions that have arisen from these studies. In this paper we answer one of these questions by applying a significant theorem of M.E. Rudin [7].

Definition 1. A family $\mathcal{F} = \{F_\alpha : \alpha \in A\}$ is *closure-preserving* if for every subset $B \subseteq A$,

$$\bigcup_{\beta \in B} \overline{F}_\beta = \overline{\bigcup_{\beta \in B} F_\beta}.$$

Likewise, $\mathcal{F}$ is *hereditarily closure-preserving* if for every $B \subseteq A$ and $\{H_\beta : \beta \in B\}$ where $H_\beta \subseteq F_\beta$, $\bigcup_{\beta \in B} \overline{H}_\beta = \overline{\bigcup_{\beta \in B} H_\beta}$.

Let P be one of the following properties; discrete (D), locally finite (LF), hereditarily closure-preserving (HCP), and closure-preserving (CP). The symbol λ will denote any countable ordinal.

Definition 2. A space X is *$B(P, \lambda)$-refinable* provided every open cover $\mathcal{U}$ of X has a refinement $\mathcal{E} = \cup \{\mathcal{E}_\beta : \beta < \lambda\}$ which satisfies i) $\{\cup \mathcal{E}_\beta : \beta < \lambda\}$ paritions X, ii) for every $\beta < \lambda$, $\mathcal{E}_\beta$ is a relatively P collection of closed subsets of the subspace $X - \cup \{\cup \mathcal{E}_\mu : \mu < \beta\}$, and iii) for every $\beta < \lambda$, $\cup \{\cup \mathcal{E}_\mu : \mu < \beta\}$ is a closed set.

The collection $\mathcal{E}$ is often called a $B(P, \lambda)$-refinement of $\mathcal{U}$.

Problem. When are the properties $B(D, \lambda)$-refinable, $B(LF, \lambda)$-refinable and $B(HCP, \lambda)$-refinable equivalent? Partial answers to this question are found in [6]. We now provide a more complete answer using the following result [7].

Theorem 1 (Rudin). *Let X be a collectionwise normal space and $\mathcal{U}$ an open cover of X. If $\mathcal{U}$ has a closed hereditarily closure-perserving refinement, then $\mathcal{U}$ has a locally finite closed refinement.*

Theorem 2 *In a collectionwise* ***normal*** *space X, the following are equivalent.*

(i) *X is paracompact.*

(ii) *X is $B(D, \lambda)$-refinable.*

(iii) *X is $B(LF, \lambda)$-refinable.*

(iv) *X is $B(HCP, \lambda)$-refinable.*

Proof. It is known (see [6]) that (i) $\equiv$ (ii) $\equiv$ (iii) and clear that (iii) $\Rightarrow$ (iv). Here we need only show that (iv) $\Rightarrow$ (i). Let $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ be an open cover of X and $\mathcal{E} = \cup \{\mathcal{E}_\beta : \beta < \lambda\}$ a $B(HCP, \lambda)$-refinement of $\mathcal{U}$. By Theorem 3 of [6] we have that X is expandable. We construct for each $\delta < \lambda$, a LF-open partial refinement $\mathcal{W}_\delta$ of $\mathcal{U}$ such that W_δ covers $(\cup \mathcal{E}_\delta) -$