50. *The Generalized Divisor Problem in Arithmetic Progressions*

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Let $d_z(n)$ be a multiplicative function defined by

$$\zeta^z(s) = \sum_{n=1}^{\infty} \frac{d_z(n)}{n^s} \quad (\sigma > 1)$$

where $s = \sigma + it$, z is a complex number, and $\zeta(s)$ is the Riemann zeta function. Here $\zeta^z(s) = \exp(z \log \zeta(s))$ and let $\log \zeta(s)$ take real values for real $s > 1$.

The following asymptotic formula was considered by G. J. Rieger [5], which is a generalization of Theorem 1 of A. Selberg [6]:

$$(1) \qquad D_z(x, q, l) \equiv \sum_{\substack{n \le x \\ n \equiv l (\mathrm{mod}\ q)}} d_z(n) = \left(\frac{\varphi(q)}{q}\right)^z \frac{x}{\Gamma(z)\varphi(q)} (\log x)^{z-1} + O\left(\left(\frac{\varphi(q)}{q}\right)^z \frac{x}{\varphi(q)} (\log x)^{\Re z - 2} \log\log 4q\right)$$

uniformly for $|z| \le A$, $q \le (\log x)^{\tau}$, $(q, l) = 1$, where A and τ are any fixed positive numbers.

Next, let $\pi_k(x)$ be the number of integers $\le x$ which are products of k distinct primes. For $k = 1$, $\pi_k(x)$ reduces to $\pi(x)$, the number of primes not exceeding x. Selberg considered $D_z(x)$ not only for its own sake but also with an intension to derive

$$(2) \qquad \pi_k(x) = \frac{xQ(\log\log x)}{\log x} + O\left(\frac{x(\log\log x)^k}{k!(\log x)^2}\right)$$

uniformly for $1 \le k \le A \log\log x$, where $Q(x)$ is polynomial of degree $k - 1$.

Now we define $\pi_k(x, q, l)$ as a generalization of $\pi_k(x)$ by

$$\pi_k(x, q, l) \equiv \sum_{\substack{n \le x \\ n \equiv l (\mathrm{mod}\ q) \\ n = p_1 \cdots p_k (p_i \ne p_j)}} 1.$$

In this paper we shall consider the connections between the asymptotic formulas of $D_z(x, q, l)$, $\pi_k(x, q, l)$ and the location of zeros of the Dirichlet L-function. In particular we shall establish some necessary and sufficient conditions for the truth of the Riemann hypothesis, so that this paper gives a generalization of [1] to arithmetic progressions.

The main term of (1) and (2) is, however, inconvenient for our aim so that we introduce the following integrals as the main terms of $D_z(x, q, l)$ and $\pi_k(x, q, l)$ respectively: