81. *Remarks on Viscosity Solutions for Evolution Equations*

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1. Introduction. We consider a degenerate parabolic equation

$$\partial u/\partial t+F(t, x, u, \nabla u, \nabla^2 u)=0, \tag{1}$$

where ∇ stands for the spatial derivatives. We are concerned with a viscosity subsolution which needs not to be continuous. We say a function $u(t,x)$ defined in a parabolic neighborhood of (t_0, x_0) is *left accessible* at (t_0, x_0) if there are sequences $x_l \to x_0$, $t_l \to t_0$ with $t_l < t_0$ such that $\lim_{l\to\infty} u(t_l, x_l) = u(t_0, x_0)$. Our goal is to show that a viscosity subsolution is left accessible at each (parabolic) interior point of the domain of definition for a wide class of F. We also clarify the relation between viscocity subsolutions defined on time interval $(0, T)$ and those on $(0, T]$. Similar problems are studied in other contexts by Crandall and Newcomb [3] and by Ishii [7]. We thank Professor Hitoshi Ishii for pointing out these references.

There are technical errors in the proof of Ishii's lemma up to the terminal time in our previous work [1, Lemma 3.1 and Proposition 3.2]. If we note left accessibility, the proof can be easily fixed. We take this opportunity to correct technical errors in [1] somewhat related to left accessibility. We thank Professor Joseph Fu for pointing out a couple of errors in the proof of [1, Lemma 3.1 and Proposition 3.2].

For $h: L \to \boldsymbol{R}$ $(L \subset \boldsymbol{R}^d)$ we associate its *lower* (*upper*) *semicontinuous relaxation* $h_*(h^*): \bar{L} \to \tilde{\boldsymbol{R}} = \boldsymbol{R} \cup \{\pm\infty\}$ defined by

$$h_*(z) = \lim_{\varepsilon\downarrow 0} \inf\{h(y)\,;\, |z-y|<\varepsilon,\, y\in L\}, \quad z \in \bar{L}$$

and $h^*(z) = -(-h)_*(z)$. Let Ω be an open set in $\boldsymbol{R}^n$. For $T>0$ let W be a dense subset of $A=(0, T]\times\Omega\times\boldsymbol{R}\times\boldsymbol{R}^n\times\boldsymbol{S}^n$, where $\boldsymbol{S}^n$ denotes the space of $n\times n$ real symmetric matrices. Suppose that $F=F(t,x,r,p,X)$ is a real valued function defined in W. Since W is dense in A, F^* and $F_*: A\to\tilde{\boldsymbol{R}}$ are well-defined. Any function $u: Q\to\boldsymbol{R}$ (resp. $Q_0\to\boldsymbol{R}$) is called a *viscosity subsolution* of (1) in $Q=(0,T]\times\Omega$ (resp. $Q_0=(0,T)\times\Omega$) if $u^*<\infty$ on $\bar{Q}$ and if, whenever $\psi\in C^2(Q)$ (resp. $C^2(Q_0)$), $(t,x)\in Q$ (resp. Q_0) and $(u^*-\psi)(t,x)=\max_Q(u^*-\psi)$ (resp. $\max_{Q_0}(u^*-\psi)$) it holds that

$$\psi_t(t,x)+F_*(t,x,u^*(t,x),\nabla\psi(t,x),\nabla^2\psi(t,x))\le 0, \tag{2}$$

where $\psi_t=\partial\psi/\partial t$. We shall suppress the word viscocity. One can easily observe that u is a subsolution of (1) in Q (resp. Q_0) if and only if u is a subsolution of (1) in $(0,T]\times U(x)$ (resp. $(0,T)\times U(x)$) for all $x\in\Omega$, where $U(x)$ is an open ball centered at x in Ω.

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